

KPZ equation, its renormalization and invariant measures

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Abstract

The Kardar-Parisi-Zhang (KPZ) equation is a stochastic partial differential equation which is ill-posed because the nonlinearity is marginally defined with respect to the roughness of the forcing noise. However, its Cole-Hopf solution, defined as the logarithm of the solution of the linear stochastic heat equation (SHE) with a multiplicative noise, is a mathematically well-defined object. In fact, Hairer [13] has recently proved that the solution of SHE can actually be derived through the Cole-Hopf transform of the solution of the KPZ equation with a suitable renormalization under periodic boundary conditions. This transformation is unfortunately not well adapted to studying the invariant measures of these Markov processes.

The present paper introduces a different type of regularization for the KPZ equation on the whole line \mathbb{R} or under periodic boundary conditions, which is appropriate from the viewpoint of studying the invariant measures. The Cole-Hopf transform applied to this equation leads to an SHE with a smeared noise having an extra complicated nonlinear term. Under time average and in the stationary regime, it is shown that this term can be replaced by a simple linear term, so that the limit equation is the linear SHE with an extra linear term with coefficient $\frac{1}{24}$. The methods are essentially stochastic analytic: The Wiener-Itô expansion and a similar method for establishing the Boltzmann-Gibbs principle are used. As a result, it is shown that the distribution of a two-sided geometric Brownian motion with a height shift given by Lebesgue measure is invariant under the evolution determined by the SHE on \mathbb{R} .

1 Introduction and main results

The Kardar-Parisi-Zhang (KPZ) equation is the stochastic partial differential equation (SPDE)

$$(1.1) \quad \partial_t h = \frac{1}{2} \partial_x^2 h + \frac{1}{2} (\partial_x h)^2 + \dot{W}(t, x), \quad x \in \mathbb{R},$$

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where $\dot{W}(t, x)$ is the space-time Gaussian white noise, in particular, it has covariance

$$E[\dot{W}(t, x)\dot{W}(s, y)] = \delta(x - y)\delta(t - s).$$

We consider (1.1) in one dimension on the whole line, or on finite intervals with periodic boundary conditions. This SPDE is used in the physics literature as a general model for the fluctuations of a growing interface and $h = h(t, x) \in \mathbb{R}$ describes the height of the interface at time t and position $x \in \mathbb{R}$. The coefficients $\frac{1}{2}$ are not essential, since one can change them under scalings in time, position and values of h . The importance of the KPZ equation comes from the fact that it reflects the behavior of a wide class of microscopic systems. Unfortunately, the equation (1.1) is ill-posed and does not make sense as written. Indeed, the linear SPDE obtained from (1.1) by dropping the nonlinear term $\frac{1}{2}(\partial_x h)^2$ has a solution $h(t, x)$ which is $(\frac{1}{2} - \varepsilon)$ -Hölder continuous for every $\varepsilon > 0$ in the spatial variable $x \in \mathbb{R}$, and this suggests that the term $\frac{1}{2}(\partial_x h)^2$ would diverge in the equation (1.1). To compensate for this diverging factor, one needs to introduce a renormalization and the correct form of the KPZ equation would be

$$(1.2) \quad \partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}\left((\partial_x h)^2 - \delta_x(x)\right) + \dot{W}(t, x), \quad x \in \mathbb{R}.$$

The delta function δ_x evaluated at x is certainly $+\infty$. At least heuristically, one can derive (1.2) from the well-defined linear stochastic heat equation (1.3) explained below by applying Cole-Hopf transform and Itô's formula and $-\frac{1}{2}\delta_x(x)$ appears as an Itô correction term; see (1.7) and (1.8) below, in which $\xi^\varepsilon \rightarrow \delta_0(0) = \delta_x(x)$, since $\eta_2^\varepsilon(x) \rightarrow \delta_0(x)$, and $\dot{W}^\varepsilon \rightarrow \dot{W}$ as $\varepsilon \downarrow 0$ at least formally. The result is called the *Cole-Hopf solution*. Recently, Hairer [13] has shown how to make sense of (1.2) (with periodic boundary conditions) by introducing a renormalization structure without passing through the Cole-Hopf transform. The resulting (unique) solutions are indeed the Cole-Hopf solutions (1.6). In this sense, and also because they arise in the weakly asymmetric limit of exclusion models [2], the Cole-Hopf solution is the physically correct solution of the KPZ equation.

To explain the Cole-Hopf solution of the KPZ equation, let us consider the following one dimensional linear stochastic heat equation (SHE) for $Z = Z(t, x)$ with a multiplicative noise:

$$(1.3) \quad \partial_t Z = \frac{1}{2}\partial_x^2 Z + Z\dot{W}(t, x), \quad x \in \mathbb{R},$$

having an initial value $Z(0, x) \geq 0$. Unlike the KPZ equation (1.1), this is a well-posed SPDE. In fact, the solution of the SPDE (1.3) is defined in a weak sense, that is, $Z(t) = \{Z(t, x); x \in \mathbb{R}\}$ is called a *solution in generalized functions' sense* if it is adapted to the increasing family of σ -fields generated by $\{\dot{W}(s, x); 0 \leq s \leq t, x \in \mathbb{R}\}$ and satisfies

$$(1.4) \quad \langle Z(t), \varphi \rangle = \langle Z(0), \varphi \rangle + \frac{1}{2} \int_0^t \langle Z(s), \partial_x^2 \varphi \rangle ds + \int_0^t \int_{\mathbb{R}} Z(s, x) \varphi(x) W(ds dx),$$

for every $\varphi \in C_0^\infty(\mathbb{R})$, where $\langle Z, \varphi \rangle = \int_{\mathbb{R}} Z(x) \varphi(x) dx$ and the last term is defined as the Itô integral with respect to the space-time Gaussian white noise. The process $Z(t)$ is called a *mild solution* if it is adapted and satisfies the stochastic integral equation

$$(1.5) \quad Z(t, x) = \int_{\mathbb{R}} p(t, x, y) Z(0, y) dy + \int_0^t \int_{\mathbb{R}} p(t - s, x, y) Z(s, y) W(ds dy),$$

where $p(t, x, y) = \frac{1}{\sqrt{2\pi t}} e^{-(y-x)^2/2t}$ is the heat kernel on \mathbb{R} . These two notions of solutions are equivalent in the class \mathcal{C}_{tem} , which consists of all $Z \in \mathcal{C} = C(\mathbb{R}, \mathbb{R})$ satisfying

$$\|Z\|_{C_r} = \sup_{x \in \mathbb{R}} e^{-r|x|} |Z(x)| < \infty,$$

for all $r > 0$, equipped with the topology induced by norms $\{\|\cdot\|_{C_r}; r > 0\}$. It is known that, if $Z(0) \in \mathcal{C}_{\text{tem}}$, the SPDE (1.3) has a unique solution, in both generalized functions' and mild senses, such that $Z(\cdot) \in C([0, \infty), \mathcal{C}_{\text{tem}})$ a.s. Moreover, the strong comparison theorem holds, that is, if $Z(0)$ satisfies in addition that $Z(0) \in \bar{\mathcal{C}}_+ = C(\mathbb{R}, [0, \infty))$ and $Z(0, x) > 0$ for some $x \in \mathbb{R}$, then $Z(t) \in C((0, \infty), \mathcal{C}_+)$ a.s., where $\mathcal{C}_+ = C(\mathbb{R}, \mathbb{R}_+)$, $\mathbb{R}_+ = (0, \infty)$, equipped with the usual topology of uniform convergence on each bounded interval; see [20] and Corollary 1.4 of [21].

The *Cole-Hopf solution* of the KPZ equation is defined from the solution of (1.3) as

$$(1.6) \quad h(t, x) := \log Z(t, x),$$

which is well-defined since $Z(t, x) > 0$. As we mentioned above, in order to link the Cole-Hopf solution to the KPZ equation, we need to deal with an infinite Itô correction term. In other words, a certain renormalization factor which balances with this diverging term should be introduced in the KPZ equation.

The simplest approximation scheme is to consider

$$(1.7) \quad \partial_t h = \frac{1}{2} \partial_x^2 h + \frac{1}{2} ((\partial_x h)^2 - \xi^\varepsilon) + \dot{W}^\varepsilon(t, x), \quad x \in \mathbb{R},$$

where $\dot{W}^\varepsilon(t, x) = (\dot{W}(t) * \eta^\varepsilon)(x)$ is a smeared noise defined from a usual symmetric convolution kernel η^ε which tends to the δ -function δ_0 as $\varepsilon \downarrow 0$ and $\xi^\varepsilon = \eta_2^\varepsilon(0)$ with $\eta_2^\varepsilon = \eta^\varepsilon * \eta^\varepsilon$; see Section 2.1 for more details. Under the Cole-Hopf transform (1.6) or $Z(t, x) := e^{h(t, x)}$, by applying Itô's formula, this is equivalent to the SPDE

$$(1.8) \quad \partial_t Z = \frac{1}{2} \partial_x^2 Z + Z \dot{W}^\varepsilon(t, x), \quad x \in \mathbb{R},$$

see [2], (3.6). It is easy to see that the solution of (1.8) converges to that of (1.3) as $\varepsilon \downarrow 0$, and therefore the solution of (1.7) converges to the Cole-Hopf solution of the KPZ equation. However, from the point of view of invariant measures, (1.7) is not a good approximation; in fact it is an open problem to describe the invariant measures of (1.7).

The present paper introduces a renormalization different from (1.7) or the one considered in [13], better adapted to finding the invariant measures. We consider the following KPZ approximating equation:

$$(1.9) \quad \partial_t h = \frac{1}{2} \partial_x^2 h + \frac{1}{2} ((\partial_x h)^2 - \xi^\varepsilon) * \eta_2^\varepsilon + \dot{W}^\varepsilon(t, x), \quad x \in \mathbb{R}.$$

It is a common fact (or folklore) that the invariant measures are essentially unchanged if we apply an operator A (in our case the convolution with η^ε) to the noise term and apply A twice to the drift term at the same time; see e.g., [14]. Here, the convolution commutes with the second derivatives so that we don't put it in the first term. Then, the

Cole-Hopf transform $Z^\varepsilon(t, x) = e^{h^\varepsilon(t, x)}$ applied to the solution $h = h^\varepsilon(t, x)$ of (1.9) leads to an SHE with a smeared noise and an extra complex nonlinear term involving a certain renormalization structure:

$$(1.10) \quad \partial_t Z = \frac{1}{2} \partial_x^2 Z + \frac{1}{2} Z \left\{ \left(\frac{\partial_x Z}{Z} \right)^2 * \eta_2^\varepsilon - \left(\frac{\partial_x Z}{Z} \right)^2 \right\} + Z \dot{W}^\varepsilon(t, x), \quad x \in \mathbb{R}.$$

One of the main contributions of this paper is to show that this nonlinear term, that is the middle term in the right hand side of (1.10), can be replaced by a simple linear term divided by a specific constant 24 in the limit when the corresponding tilt process is in equilibrium; see Theorems 3.3 and 3.11 below. Thus, we derive the SPDE

$$(1.11) \quad \partial_t Z = \frac{1}{2} \partial_x^2 Z + \frac{1}{24} Z + Z \dot{W}(t, x), \quad x \in \mathbb{R},$$

in the limit as $\varepsilon \downarrow 0$. Or, we can rephrase it, that the solution $h^\varepsilon(t, x)$ of (1.9) converges to $h(t, x) + \frac{1}{24}t$, where $h(t, x)$ is the Cole-Hopf solution defined by (1.6). The constant $\frac{1}{24}$ frequently appears in KPZ related papers and describes the speed of the vertical drift of the interface. The same constant also appears in our formulation and this provides it with a probabilistic meaning. If the convolution kernel η^ε is asymmetric and satisfies a certain condition, a constant different from $\frac{1}{24}$ appears in the limit; see Remark 3.3-(2).

For technical reasons (see Lemma 3.12 and Proposition 3.13), in order to study the limit $\varepsilon \downarrow 0$, we need to work with the KPZ approximating equation (1.9) on finite intervals with periodic boundary conditions:

$$(1.12) \quad \partial_t h = \frac{1}{2} \partial_x^2 h + \frac{1}{2} ((\partial_x h)^2 - \xi^\varepsilon) * \eta_2^\varepsilon + \dot{W}^\varepsilon(t, x), \quad x \in \mathbb{S}_M,$$

where $\mathbb{S}_M = \mathbb{R}/M\mathbb{Z} (= [0, M))$, $M \geq 1$, is a continuous one-dimensional torus of size M . The convolution $*\eta_2^\varepsilon$ is defined in a periodic sense. A similar SPDE was studied in Da Prato et. al. [5]. The Cole-Hopf transformed process $Z^{\varepsilon, M}(t, x) = e^{h^{\varepsilon, M}(t, x)}$, $x \in \mathbb{S}_M$ applied to the solution $h = h^{\varepsilon, M}(t, x)$ of (1.12) satisfies the SPDE:

$$(1.13) \quad \partial_t Z = \frac{1}{2} \partial_x^2 Z + \frac{1}{2} Z \left\{ \left(\frac{\partial_x Z}{Z} \right)^2 * \eta_2^\varepsilon - \left(\frac{\partial_x Z}{Z} \right)^2 \right\} + Z \dot{W}^\varepsilon(t, x), \quad x \in \mathbb{S}_M.$$

We also consider the SPDE

$$(1.14) \quad \partial_t Z = \frac{1}{2} \partial_x^2 Z + \frac{1}{24} Z + Z \dot{W}(t, x), \quad x \in \mathbb{S}_M,$$

which will appear in the limit as $\varepsilon \downarrow 0$.

To state our first result, we introduce some notation. Let $B^M = \{B^M(x); x \in \mathbb{S}_M\}$ be the pinned Brownian motion satisfying $B^M(0) = B^M(M) = 0$. Let ν^M and $\nu^{\varepsilon, M}$ be the distributions of B^M and $\{B^M * \eta^\varepsilon(x) - B^M * \eta^\varepsilon(0); x \in \mathbb{S}_M\}$ with the convolution defined in a periodic sense on the space $\mathcal{C}_M = C(\mathbb{S}_M) (= C(\mathbb{S}_M, \mathbb{R}))$, respectively. For a random continuous function $h = \{h(x); x \in \mathbb{S}_M\}$ on \mathbb{S}_M , we call $\nabla h = \{h(x) - h(y); x, y \in \mathbb{R}\}$ for h periodically extended on \mathbb{R} the tilt variables of h and denote $\nabla h \stackrel{\text{law}}{=} \nu^{\varepsilon, M}$ (or ν^M) if the law of $\{h(x) - h(0); x \in \mathbb{S}_M\}$ is given by $\nu^{\varepsilon, M}$ (or ν^M). In fact, $\nu^{\varepsilon, M}$ is invariant for the tilt process determined from the SPDE (1.12); see Theorem 2.1-(1).

Our convergence result on \mathbb{S}_M is now formulated as follows.

Theorem 1.1. *We fix $M \geq 1$ and assume that the law of $Z^{\varepsilon, M}(0, \cdot) = e^{h^{\varepsilon, M}(0, \cdot)}$ is determined by $(h^{\varepsilon, M}(0, 0), \nabla h^{\varepsilon, M}(0)) \stackrel{\text{law}}{=} \delta_{h_0} \otimes \nu^{\varepsilon, M}$ with some $h_0 \in \mathbb{R}$. Then, for every $t \geq 0$, the law of the solution $Z^{\varepsilon, M}(t, \cdot)$ of the SPDE (1.13) on the space \mathcal{C}_M weakly converges as $\varepsilon \downarrow 0$ to that of the solution $Z^M(t, \cdot)$ of the SPDE (1.14) with the initial distribution determined by $Z^M(0, \cdot) = e^{h^M(0, \cdot)}$ such that $(h^M(0, 0), \nabla h^M(0)) \stackrel{\text{law}}{=} \delta_{h_0} \otimes \nu^M$.*

This theorem immediately implies that the distribution of the tilt variables of the logarithm of the solution $Z^M(t, \cdot)$ of the SPDE (1.14) is given by ν^M for every $t \geq 0$.

We next pass to the limit $M \rightarrow \infty$ by extending $Z^M(t, x)$, $x \in \mathbb{S}_M$ and ν^M periodically on \mathbb{R} . We need some more notation. Let $B = \{B(x); x \in \mathbb{R}\}$ be the two-sided Brownian motion satisfying $B(0) = 0$, that is, $\{B(x); x \geq 0\}$ and $\{B(-x); x \geq 0\}$ are both Brownian motions with time parameter $x \geq 0$ and mutually independent. Let ν and ν^ε be the distributions of B and $\{B * \eta^\varepsilon(x) - B * \eta^\varepsilon(0); x \in \mathbb{R}\}$ on the space \mathcal{C} , respectively. The tilt variables ∇h of h on \mathbb{R} are similarly defined as above and denote $\nabla h \stackrel{\text{law}}{=} \nu$ (or ν^ε) if the law of $\{h(x) - h(0); x \in \mathbb{R}\}$ is given by ν (or ν^ε). We introduce the weighted L^2 -space $L_r^2(\mathbb{R})$, $r > 0$, which is a family of all measurable functions u on \mathbb{R} such that

$$(1.15) \quad \|u\|_{L_r^2} := \left(\int_{\mathbb{R}} u(x)^2 e^{-2r\chi(x)} dx \right)^{1/2} < \infty,$$

where $\chi \in C^\infty(\mathbb{R})$ is a fixed function satisfying $\chi(x) = |x|$ for $|x| \geq 1$.

Then, it is standard to show that $Z^M(t, x)$, $x \in \mathbb{R}$ converges to the solution of the SPDE (1.11) weakly on $C([0, \infty), L_r^2(\mathbb{R}))$, $r > 0$ and ν^M (periodically extended on \mathbb{R}) converges to ν weakly on $L_r^2(\mathbb{R})$, $r > 1$ as $M \rightarrow \infty$; see Proposition 3.17 and its consequences.

As a byproduct, though the factor $\frac{1}{24}Z$ is different in (1.11), we can investigate the invariant measures of the SPDE (1.3) on \mathbb{R} . Let μ^c , $c \in \mathbb{R}$ be the distribution of $e^{B(x)+cx}$, $x \in \mathbb{R}$ on \mathcal{C}_+ , where $B(x)$ is the two-sided Brownian motion such that $\mu^c(B(0) \in dx) = dx$. In particular, μ^c are not probability measures. Our second result is that μ^c are invariant for the process $Z(t)$, which is a solution of the SPDE (1.3).

Theorem 1.2. *For every bounded, integrable and continuous function G on \mathcal{C}_+ , we have that*

$$\int_{\mathcal{C}_+} G(Z(t)) d\mu^c = \int_{\mathcal{C}_+} G(Z(0)) d\mu^c,$$

for all $t \geq 0$ and $c \in \mathbb{R}$, where the integrals in both sides are defined under the condition that $Z(0)$ is distributed under μ^c . More precisely, for example, the left hand side is given by

$$\int_{\mathcal{C}_+} E_Z[G(Z(t))] \mu^c(dZ),$$

where $E_Z[\cdot]$ stands for the expectation with respect to the solution of the SPDE (1.3) with an initial value $Z(0) = Z \in \mathcal{C}_+$.

At the level of the process $\partial_x \log Z(t, x)$, where the corresponding invariant measure is a white noise, the result was derived earlier by Bertini and Giacomin [2] via the weakly asymmetric limit of simple exclusion processes.

Note that the measure μ^c is invariant, but not reversible for the process $Z(t)$. Only a kind of Yaglom reversibility holds; see Remark 2.4 below.

If $Z(t, x)$ is a solution of (1.3) and $c \in \mathbb{R}$ then $Z^c(t, x) := e^{cx + \frac{1}{2}c^2t}Z(t, x + ct)$ is also a solution (with a new space-time Gaussian white noise \dot{W}). Therefore, if one can show the invariance of μ^0 for $Z(t)$, since μ^0 is invariant under both shifts $Z(x) \mapsto Z(x + a)$ and $Z(x) \mapsto e^a Z(x)$, we have that the μ^c is also invariant for the process $Z(t)$. For this reason, in the proof of Theorem 1.2, we may assume $c = 0$ without loss of generality and write μ for μ^0 .

We will mostly work with tilt variables $\partial_x h$ or $\partial_x \log Z$ rather than heights to avoid the difficulty caused by the non-normalizability of the measure μ . This is carried out by introducing an equivalence relation to the state space \mathcal{C}_+ of $Z(t, x)$; see Section 3.1. Or, one can say that we are only interested in the shapes of height functions $h(t, x) = \log Z(t, x)$ by identifying its vertical translations: $h(t, x) \sim h(t, x) + c$ for all $c \in \mathbb{R}$; see Remark 2.1 below.

Remark 1.1. *One expects μ^c , $c \in \mathbb{R}$ to be all the extremal invariant measures (except constant multipliers) for the process $Z(t)$ as in [12], but we will not investigate this here.*

The paper is organized as follows. In Section 2, we consider the KPZ approximating equations (1.9) and (1.12), and study their invariant measures; see Theorems 2.1 and 2.11. This is accomplished by introducing finite dimensional approximations due to the discretization in space, and then taking limits. In Section 3, we consider its Cole-Hopf transform and pass to the limit. We need to replace a complicated nonlinear term by a simple linear term in the limit; see Theorems 3.3 and 3.11. This procedure has a similarity to the so-called Boltzmann-Gibbs principle [16], [17], which plays an important role in establishing the equilibrium fluctuation limits for large scale interacting systems. We rely on Wiener-Itô expansion.

2 KPZ approximating equation

This section studies the invariant measures of the KPZ approximating equations (1.9) and (1.12). For this purpose, we first consider the associated tilt process $u = \partial_x h$, which satisfies the SPDEs (2.3) and (2.4) stated below, and introduce its finite dimensional approximations due to the spatial discretization. Indeed, in view of finding invariant measures, it is important to choose a special discretization scheme. The infinitesimal invariance is shown in Lemma 2.2. Then, we find the invariant measures of the SPDEs (2.3) and (2.4) by passing to the limits; see Theorem 2.1. This part is standard, especially because $0 < \varepsilon < 1$ is fixed and the noises in (2.3), (2.4) or (1.9), (1.12) are smooth. The results are the same even if we replace ξ^ε by any other constants.

One can actually check the infinitesimal invariance directly for (2.3) without introducing the spatial discretization; see Remark 2.3. The reason we do not do it that way is that there are not clear enough results in the infinite dimensional setting telling us that the infinitesimal invariance implies the global invariance.

2.1 Approximating equations and invariant measures for tilt processes

Let $\eta \in C_0^\infty(\mathbb{R})$ be a function satisfying $\eta(x) \geq 0$, $\eta(x) = \eta(-x)$ and $\int_{\mathbb{R}} \eta(x) dx = 1$. We set $\eta^\varepsilon(x) = \eta(x/\varepsilon)/\varepsilon$ for $\varepsilon > 0$, $\eta_2(x) = \eta * \eta(x)$, and $\eta_2^\varepsilon(x) = \eta_2(x/\varepsilon)/\varepsilon$. Note that $\eta_2^\varepsilon(x) = \eta^\varepsilon * \eta^\varepsilon(x)$. To fix ideas, we assume that $\text{supp } \eta \subset [-1, 1]$, so that $\text{supp } \eta^\varepsilon \subset [-\varepsilon, \varepsilon]$ and $\text{supp } \eta_2^\varepsilon \subset [-2\varepsilon, 2\varepsilon]$. Define the smeared noise:

$$(2.1) \quad W^\varepsilon(t, x) = \langle W(t), \eta^\varepsilon(x - \cdot) \rangle,$$

and consider the KPZ approximating equation (1.9) on \mathbb{R} for $h = h^\varepsilon(t, x)$. By the symmetry of η , we have that

$$(2.2) \quad \xi^\varepsilon = \int_{\mathbb{R}} \eta^\varepsilon(y)^2 dy (= \eta_2^\varepsilon(0)),$$

in (1.9). The solution h of the SPDE (1.9) is smooth in x and we are concerned with the associated tilt process $u = \partial_x h$, which satisfies the stochastic Burgers' equation:

$$(2.3) \quad \partial_t u = \frac{1}{2} \partial_x^2 u + \frac{1}{2} \partial_x (u^2 * \eta_2^\varepsilon) + \partial_x \dot{W}^\varepsilon(t, x), \quad x \in \mathbb{R}.$$

In this respect, (1.9) is a kind of stochastic Hamilton-Jacobi equation. Similarly, the tilt process $u = \partial_x h$ of the solution h of the SPDE (1.12) on \mathbb{S}_M , $M \geq 1$, satisfies

$$(2.4) \quad \partial_t u = \frac{1}{2} \partial_x^2 u + \frac{1}{2} \partial_x (u^2 * \eta_2^\varepsilon) + \partial_x \dot{W}^\varepsilon(t, x), \quad x \in \mathbb{S}_M.$$

Note that $\int_{\mathbb{S}_M} u(t, x) dx = 0$ holds for (2.4).

Let ν^ε be the distribution of $\partial_x(B * \eta^\varepsilon(x))$ on \mathcal{C} , where B is the two-sided Brownian motion satisfying $B(0) = 0$; we abuse the notation for ν^ε compared with that introduced in Section 1, since the meanings are clear. Note that ν^ε is a probability measure which is independent of the choice of the value of $B(0)$. Similarly, $\nu^{\varepsilon, M}$ is the distribution of $\partial_x(B^M * \eta^\varepsilon(x))$ on $\mathcal{C}_{M,0}$, where B^M is the pinned Brownian motion, $\mathcal{C}_{M,0} = \{u \in \mathcal{C}_M; \int_{\mathbb{S}_M} u(x) dx = 0\}$ and recall $\mathcal{C}_M = C(\mathbb{S}_M)$. Then, the first main result of this section is formulated as in the following theorem. This will be extended to the height process h in Section 2.7; see Theorem 2.11.

Theorem 2.1. (1) *The probability measure $\nu^{\varepsilon, M}$ on $\mathcal{C}_{M,0}$ is invariant under (2.4), that is, for the tilt process $\partial_x h$ of the solution h of the SPDE (1.12).*
(2) *The probability measure ν^ε on \mathcal{C} is invariant under (2.3), that is, for the tilt process determined from the SPDE (1.9).*

2.2 Invariant measure of KPZ approximating equation on a discrete torus

In this section, we introduce the KPZ approximating equation on a discrete torus $\mathbb{T}_N = \{1, 2, \dots, N\}$ with periodic boundary condition. To study its invariant measure, it is important to choose a special discretization scheme as we will explain. Let $\alpha : \mathbb{Z} \rightarrow [0, \infty)$ be given and satisfy the conditions $\alpha(i) = \alpha(-i)$ and $\alpha(i) = 0$ for $i : |i| \geq K$ with some

$K \geq 1$. We naturally regard α as a function on \mathbb{T}_N assuming that N is sufficiently large compared with the size of the support of α : $N > 2K$.

For $h = (h(i))_{i \in \mathbb{T}_N} \in \mathbb{R}^{\mathbb{T}_N}$, we define $\Delta h \in \mathbb{R}^{\mathbb{T}_N}$ by $\Delta h(i) = h(i+1) + h(i-1) - 2h(i)$, $i \in \mathbb{T}_N$ and two functions $G_1(h) = (G_1(i, h))_{i \in \mathbb{T}_N}$, $G_2(h) = (G_2(i, h))_{i \in \mathbb{T}_N}$ by

$$\begin{aligned} G_1(i, h) &= (h_{i+1} - h_i)^2 + (h_i - h_{i-1})^2, \\ G_2(i, h) &= (h_{i+1} - h_i)(h_i - h_{i-1}), \quad i \in \mathbb{T}_N, \end{aligned}$$

respectively. We sometimes write h_i for $h(i)$. These are discrete analogues of $2(\partial_x h)^2$ and $(\partial_x h)^2$, respectively. For functions β, γ on \mathbb{T}_N , we define the convolution $\beta * \gamma$ on \mathbb{T}_N by $(\beta * \gamma)(i) = \sum_{k \in \mathbb{T}_N} \beta(i - k) \gamma(k)$, $i \in \mathbb{T}_N$, where $i - k$ is defined in modulo N .

We consider the stochastic differential equation for $h_t = (h_t(i))_{i \in \mathbb{T}_N} \in \mathbb{R}^{\mathbb{T}_N}$:

$$(2.5) \quad dh_t(i) = \frac{\lambda_1}{2} \Delta h_t(i) dt + \lambda_2 \{ \alpha_2 * G_1(i, h_t) + \alpha_2 * G_2(i, h_t) \} dt + \lambda_3 dw_t^\alpha(i), \quad i \in \mathbb{T}_N,$$

where $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ are arbitrary constants, $\alpha_2 = \alpha * \alpha$, $w_t^\alpha = \alpha * w_t$ and $w_t = (w_t(i))_{i \in \mathbb{T}_N}$ is a family of independent Brownian motions. We consider three operators on $\mathbb{R}^{\mathbb{T}_N}$:

$$\begin{aligned} \mathcal{L}_0^\alpha f(h) &= \frac{\lambda_1}{2} \sum_{i \in \mathbb{T}_N} \Delta h(i) \frac{\partial f}{\partial h_i} + \frac{\lambda_3^2}{2} \sum_{i, j \in \mathbb{T}_N} \alpha_2(i - j) \frac{\partial^2 f}{\partial h_i \partial h_j}, \\ \mathcal{A}_1^\alpha f(h) &= \sum_{i \in \mathbb{T}_N} (\alpha_2 * G_1)(i, h) \frac{\partial f}{\partial h_i}, \\ \mathcal{A}_2^\alpha f(h) &= \sum_{i \in \mathbb{T}_N} (\alpha_2 * G_2)(i, h) \frac{\partial f}{\partial h_i}, \end{aligned}$$

for $f \in C^2(\mathbb{R}^{\mathbb{T}_N})$. Then, $\mathcal{L}^\alpha = \mathcal{L}_0^\alpha + \lambda_2 \mathcal{A}_1^\alpha + \lambda_2 \mathcal{A}_2^\alpha$ is the generator of the SDE (2.5).

Let $\alpha^{-1} = \alpha_N^{-1}$ be the inverse matrix of $\alpha = \{\alpha(i - j)\}_{i, j \in \mathbb{T}_N}$. Note that the matrix α may not be invertible in general, but we can always make $\det \alpha \neq 0$ by slightly perturbing α and we consider such α . Let $\mu_N(dh) = e^{-I_N^\alpha(h)} dh$ be an infinite measure on $\mathbb{R}^{\mathbb{T}_N}$, where $dh = \prod_{i \in \mathbb{T}_N} dh(i)$ and

$$I_N^\alpha(h) = \frac{\lambda}{2} \sum_{j \in \mathbb{T}_N} \{ \alpha^{-1} * h(j+1) - \alpha^{-1} * h(j) \}^2, \quad \lambda = \frac{\lambda_1}{\lambda_3^2}.$$

Lemma 2.2. *For every $f, g \in C_b^2(\mathbb{R}^{\mathbb{T}_N})$, we have that*

$$(2.6) \quad \int g(h) \mathcal{L}_0^\alpha f(h) d\mu_N = \int f(h) \mathcal{L}_0^\alpha g(h) d\mu_N.$$

In particular, $\int \mathcal{L}_0^\alpha f(h) d\mu_N = 0$. We also have that

$$(2.7) \quad \int \mathcal{A}_1^\alpha f(h) d\mu_N = - \int \mathcal{A}_2^\alpha f(h) d\mu_N.$$

Accordingly, we have that

$$(2.8) \quad \int \mathcal{L}^\alpha f(h) d\mu_N = 0.$$

Proof. We first compute derivatives of I_N^α :

$$\begin{aligned}
(2.9) \quad & \frac{\partial}{\partial h_i} I_N^\alpha(h) \\
&= \lambda \sum_{j \in \mathbb{T}_N} \{ \alpha^{-1} * h(j+1) - \alpha^{-1} * h(j) \} \frac{\partial}{\partial h_i} \{ \alpha^{-1} * h(j+1) - \alpha^{-1} * h(j) \} \\
&= \lambda \sum_{j \in \mathbb{T}_N} \sum_{k \in \mathbb{T}_N} \{ \alpha^{-1}(j+1-k) - \alpha^{-1}(j-k) \} h(k) \cdot \{ \alpha^{-1}(j+1-i) - \alpha^{-1}(j-i) \} \\
&= \lambda \sum_{k \in \mathbb{T}_N} \{ 2\alpha_2^{-1}(i-k) - \alpha_2^{-1}(i+1-k) - \alpha_2^{-1}(i-1-k) \} h(k) \\
&= -\lambda \sum_{k \in \mathbb{T}_N} \alpha_2^{-1}(i-k) \Delta h(k) = -\lambda (\alpha_2^{-1} * \Delta h)(i).
\end{aligned}$$

We now prove the symmetry (2.6) of \mathcal{L}_0^α . To this end,

$$\begin{aligned}
\int g \frac{\partial^2 f}{\partial h_i \partial h_j} d\mu_N &= - \int \frac{\partial}{\partial h_j} \left(g e^{-I_N^\alpha(h)} \right) \frac{\partial f}{\partial h_i} dh \\
&= - \int \left(\frac{\partial g}{\partial h_j} - g \frac{\partial I_N^\alpha}{\partial h_j} \right) \frac{\partial f}{\partial h_i} d\mu_N \\
&= - \int \left(\frac{\partial g}{\partial h_j} + g \lambda (\alpha_2^{-1} * \Delta h)(j) \right) \frac{\partial f}{\partial h_i} d\mu_N,
\end{aligned}$$

by (2.9). Therefore, we have that

$$\begin{aligned}
\int g \mathcal{L}_0^\alpha f d\mu_N &= -\frac{\lambda_3^2}{2} \sum_{i,j} \int \alpha_2(i-j) \frac{\partial g}{\partial h_j} \frac{\partial f}{\partial h_i} d\mu_N + \frac{\lambda_1}{2} \sum_i \int g \Delta h(i) \frac{\partial f}{\partial h_i} d\mu_N \\
&\quad - \frac{\lambda_3^2}{2} \cdot \lambda \sum_{i,j} \int g \alpha_2(i-j) (\alpha_2^{-1} * \Delta h)(j) \frac{\partial f}{\partial h_i} d\mu_N \\
&= -\frac{\lambda_3^2}{2} \sum_{i,j} \int \alpha_2(i-j) \frac{\partial g}{\partial h_j} \frac{\partial f}{\partial h_i} d\mu_N.
\end{aligned}$$

This shows (2.6). We next prove (2.7). For $\ell = 1, 2$,

$$\begin{aligned}
\int \mathcal{A}_\ell^\alpha f(h) d\mu_N &= - \sum_i \int f \frac{\partial}{\partial h_i} \left\{ (\alpha_2 * G_\ell)(i) e^{-I_N^\alpha(h)} \right\} dh \\
&= - \int f \left\{ \sum_i \frac{\partial}{\partial h_i} (\alpha_2 * G_\ell)(i) - \sum_i (\alpha_2 * G_\ell)(i) \frac{\partial I_N^\alpha}{\partial h_i} \right\} d\mu_N.
\end{aligned}$$

Here, noting that $\frac{\partial G_\ell}{\partial h_i}(j) = 0$ if $j \neq i, i \pm 1$, the first sum vanishes both for $\ell = 1, 2$:

$$\begin{aligned}
& \sum_i \frac{\partial}{\partial h_i} (\alpha_2 * G_\ell)(i) \\
&= \sum_i \frac{\partial}{\partial h_i} \{ \alpha_2(-1) G_\ell(i+1) + \alpha_2(0) G_\ell(i) + \alpha_2(1) G_\ell(i-1) \}
\end{aligned}$$

$$= \begin{cases} -2(\alpha_2(0) + \alpha_2(1)) \sum_i \Delta h(i) = 0, & \text{for } \ell = 1, \\ \alpha_2(0) \sum_i \Delta h(i) + \alpha_2(1) \sum_i \{-(h_{i+2} - h_{i+1}) + (h_{i-1} - h_{i-2})\} = 0, & \text{for } \ell = 2. \end{cases}$$

On the other hand, from (2.9), the second sum can be rewritten as

$$-\sum_i (\alpha_2 * G_\ell)(i) \frac{\partial I_N^\alpha}{\partial h_i} = \lambda \sum_i (\alpha_2 * G_\ell)(i) (\alpha_2^{-1} * \Delta h)(i) = \lambda \sum_i G_\ell(i) \Delta h(i),$$

for $\ell = 1, 2$. However,

$$\begin{aligned} & \sum_i G_1(i) \Delta h(i) \\ &= \sum_i \{(h_{i+1} - h_i)^2 + (h_i - h_{i-1})^2\} \{(h_{i+1} - h_i) - (h_i - h_{i-1})\} \\ &= \sum_i (h_{i+1} - h_i)^3 - \sum_i (h_{i+1} - h_i)^2 (h_i - h_{i-1}) + \sum_i (h_i - h_{i-1})^2 (h_{i+1} - h_i) - \sum_i (h_i - h_{i-1})^3 \\ &= \sum_i G_2(i) \{(h_i - h_{i-1}) - (h_{i+1} - h_i)\} \\ &= -\sum_i G_2(i) \Delta h(i). \end{aligned}$$

This implies (2.7). (2.8) is immediate from (2.6) and (2.7). \square

We can apply Echeverria's result [7] for the finite dimensional SDE (2.5) and Lemma 2.2 proves the invariance of μ_N .

We define the tilt variables $u = (u(i))_{i \in \mathbb{T}_N}$ associated with $h = (h(i))_{i \in \mathbb{T}_N}$ by $u(i) = \nabla h(i) := h(i+1) - h(i)$. Note that u always satisfies $\sum_{i \in \mathbb{T}_N} u(i) = 0$, and $I_N^\alpha(h) = \tilde{I}_N^\alpha(u) := \frac{\lambda}{2} \sum_j \{\alpha^{-1} * u(j)\}^2$. Let $\nu_N(du) = e^{-\tilde{I}_N^\alpha(u)} du / Z_N^\alpha$ be a probability measure on $\mathbb{R}_0^{\mathbb{T}_N} := \{u \in \mathbb{R}^{\mathbb{T}_N}; \sum_{i \in \mathbb{T}_N} u(i) = 0\}$, where du is the Lebesgue measure on this space and Z_N^α is a normalizing constant. Then, our result for the height process h can be transformed into that for the tilt process u :

Proposition 2.3. *The probability measure ν_N on $\mathbb{R}_0^{\mathbb{T}_N}$ is invariant for the tilt process $u = \nabla h$ of the SDE (2.5).*

Note that we will later consider $u^N \equiv \nabla_N h = N \nabla h$; see Lemma 2.5 and (2.11) below.

2.3 Invariant measure of KPZ approximating equation on a continuous torus

Under a proper scaling in space $i \mapsto x = i/N$, parameters $\lambda_1, \lambda_2, \lambda_3$ and $\alpha(\cdot)$, one can show that the stationary solution of (2.5) converges weakly to that of the SPDE (1.12) with $M = 1$, i.e., (1.12) for $x \in \mathbb{S} = \mathbb{R}/\mathbb{Z} (= [0, 1))$, or, for the corresponding tilt process, to the SPDE (2.4) with $M = 1$ for fixed $\varepsilon > 0$, by showing the tightness of the sequence of stationary solutions of the SDE (2.5). The goal is to show the following proposition, whose proof will be completed in Section 2.5. This proposition proves Theorem 2.1-(1).

Proposition 2.4. *The probability measure $\nu^{\varepsilon,1}$ on $\mathcal{C}_{1,0}$ is invariant under the SPDE (2.4) with $M = 1$. By a simple scaling argument, we see that the probability measure $\nu^{\varepsilon,M}$ on $\mathcal{C}_{M,0}$ is invariant under the SPDE (2.4).*

We first show the convergence of the invariant measure. For $u \in \mathbb{R}_0^{\mathbb{T}_N}$, we define $u^N = \{u^N(x); x \in \mathbb{S}\}$ by a linear interpolation of $\{u^N(\frac{i}{N}) := Nu(i)\}_{i \in \mathbb{T}_N}$, that is

$$(2.10) \quad \begin{aligned} u^N(x) &= u^N(\frac{i+1}{N}) \cdot N(x - \frac{i}{N}) + u^N(\frac{i}{N}) \cdot N(\frac{i+1}{N} - x) \\ &= N^2 u(i+1)(x - \frac{i}{N}) + N^2 u(i)(\frac{i+1}{N} - x), \quad x \in [\frac{i}{N}, \frac{i+1}{N}). \end{aligned}$$

Lemma 2.5. *Consider ν_N on $\mathbb{R}_0^{\mathbb{T}_N}$ by choosing $\alpha : \alpha(i) = \frac{1}{N}\eta^\varepsilon(\frac{i}{N})$ and $\lambda = N$. Then, as $N \rightarrow \infty$, the distribution of u^N under ν_N weakly converges to $\nu^{\varepsilon,1}$ on the space $\mathcal{C}_1 = C(\mathbb{S})$.*

Proof. We first observe that the law of $\{\nabla_N(\alpha * B^1(\frac{\cdot}{N}))(i)\}$ coincides with that of $\{u^N(\frac{i}{N}) = Nu(i)\}_{i \in \mathbb{T}_N}$ under ν_N , where $\{B^1(x); x \in \mathbb{S}\}$ is the pinned Brownian motion such that $B^1(0) = B^1(1) = 0$. In fact, for every $f \in C_b(\mathbb{R}_0^{\mathbb{T}_N})$,

$$\begin{aligned} E^{\nu_N}[f(u)] &= \frac{1}{Z_N^\alpha} \int_{\mathbb{R}_0^{\mathbb{T}_N}} f(u) e^{-\frac{N}{2} \sum_j \{\alpha^{-1} * u(j)\}^2} du \\ &= \frac{1}{\tilde{Z}_N^\alpha} \int_{\mathbb{R}_0^{\mathbb{T}_N}} f(\alpha * \tilde{u}) e^{-\frac{N}{2} \sum_j \tilde{u}(j)^2} d\tilde{u} \\ &= E[f(\alpha * \nabla B^1(\frac{\cdot}{N}))], \end{aligned}$$

where we have applied the change of variables: $u = \alpha * \tilde{u}$, that is, $\tilde{u} = \alpha^{-1} * u$ and $d\tilde{u} = C_N du$ for the second line and note that the distribution of $\{\tilde{u}(j)\}$ under the probability measure $e^{-\frac{N}{2} \sum_j \tilde{u}(j)^2} d\tilde{u} / \tilde{Z}_N^\alpha$ is equal to that of $\{(\nabla B^1(\frac{\cdot}{N}))(j)\}$ for the third line. Since $\alpha * \nabla B^1(\frac{\cdot}{N}) = \nabla(\alpha * B^1(\frac{\cdot}{N}))$, the above computation implies that the law of $\{\nabla_N(\alpha * B^1(\frac{\cdot}{N}))(i)\}$ is equal to that of $\{u^N(\frac{i}{N}) = Nu(i)\}$ under ν_N . However, it is easy to see that the linear interpolation of $\{\nabla_N(\alpha * B^1(\frac{\cdot}{N}))(i)\}$ converges in $C(\mathbb{S})$ to $\{\partial_x(\eta^\varepsilon * B^1)\}$ as $N \rightarrow \infty$ a.s., and this completes the proof. \square

We choose $\lambda_1 = N^2, \lambda_2 = \frac{1}{6}N^2, \lambda_3 = \sqrt{N}$ and $\alpha(i) = \frac{1}{N}\eta^\varepsilon(\frac{i}{N})$ in the SDE (2.5), and set

$$U_N(t) = \frac{1}{N} \sum_{i \in \mathbb{T}_N} \{u_t(i)^2 + (\nabla_N u_t(i))^2\},$$

where $u_t(i) := \nabla_N h_t(i) = N(h_t(i+1) - h_t(i))$. Note that $u_t = \{u_t(i)\}$ satisfies the following SDE:

$$(2.11) \quad \begin{aligned} du_t(i) &= \frac{1}{2} \Delta_N u_t(i) dt + \frac{1}{6} \nabla_N [\alpha_2 * \{u_t(\cdot)^2 + u_t(\cdot - 1)^2 + u_t(\cdot) u_t(\cdot - 1)\}](i) dt \\ &\quad + \sqrt{N} \nabla_N dw_t^\alpha(i), \end{aligned}$$

where $\Delta_N u(i) := N^2 \Delta u(i)$. We define $\{u_t^N(x); x \in \mathbb{S}\}$ by the linear interpolation of $\{u_t^N(\frac{i}{N}) := u_t(i)\}_{i \in \mathbb{T}_N}$ as in (2.10). Then $c_1 \|u_t^N\|_{H^1(\mathbb{S})}^2 \leq U_N(t) \leq c_2 \|u_t^N\|_{H^1(\mathbb{S})}^2$ with some $0 < c_1$ and $c_2 < \infty$. We denote Sobolev spaces of order $s \geq 0$ on \mathbb{S} by $H^s(\mathbb{S})$.

We consider the stationary solution of (2.11), that is, the initial value before scaling is taken as $\{u_0(i)/N\}_{i \in \mathbb{T}_N} \stackrel{\text{law}}{=} \nu_N$.

Lemma 2.6. (1) For every $T > 0$, we have the uniform bound:

$$\sup_{N \in \mathbb{N}} E \left[\sup_{0 \leq t \leq T} U_N(t) \right] < \infty.$$

(2) For every $T > 0, \varphi \in C^\infty(\mathbb{S})$ and $0 \leq s < t \leq T$,

$$E[\langle u_t^N - u_s^N, \varphi \rangle_{\mathbb{S}}^4] \leq C(\varphi)(t - s)^2,$$

holds with $C(\varphi) = C_T(\varphi) > 0$, where $\langle u^N, \varphi \rangle_{\mathbb{S}} = \int_{\mathbb{S}} u^N(x) \varphi(x) dx$.

(3) In particular, $\{u_t^N\}_{N \in \mathbb{N}}$ is tight on $C([0, T], C(\mathbb{S}))$ for every $T > 0$.

Proof. The tightness on $C([0, T], H^s(\mathbb{S}))$ with $s < 1$ follows from (1) and (2) noting that the embedding $H^1(\mathbb{S}) \subset H^s(\mathbb{S})$ is compact by Rellich's theorem; see, e.g., the proof of Proposition 3.1 in [9]. Therefore, (3) follows by noting $H^s(\mathbb{S}) \subset C(\mathbb{S})$ continuously embedded if $s > 1/2$.

To show (1), we apply Itô's formula to see that

$$\begin{aligned} dU_N(t) &= \frac{1}{N} \sum_i \{2u_t(i) du_t(i) + (du_t(i))^2 + 2\nabla_N u_t(i) d\nabla_N u_t(i) + (d\nabla_N u_t(i))^2\} \\ &= \frac{1}{N} \sum_i [u_t(i) \{ \Delta_N u_t(i) + \frac{1}{3} \nabla_N [\alpha_2 * \{u_t(\cdot)^2 + u_t(\cdot - 1)^2 + u_t(\cdot) u_t(\cdot - 1)\}](i) \} \\ &\quad + \nabla_N u_t(i) \{ \nabla_N \Delta_N u_t(i) + \frac{1}{3} \nabla_N^2 [\alpha_2 * \{u_t(\cdot)^2 + u_t(\cdot - 1)^2 + u_t(\cdot) u_t(\cdot - 1)\}](i) \} \\ &\quad + (-N \nabla_N^2 \alpha_2(0) + N \nabla_N^4 \alpha_2(0))] \cdot dt \\ &\quad + \frac{\sqrt{N}}{N} \sum_i \{2u_t(i) \nabla_N dw_t^\alpha(i) + 2\nabla_N u_t(i) \nabla_N^2 dw_t^\alpha(i)\} \\ &=: b_N(t) dt + dm_N(t), \end{aligned}$$

where $m_N(t)$ denotes the martingale part. Therefore, we have that

$$(2.12) \quad E[\sup_{0 \leq t \leq T} U_N(t)] \leq E[U_N(0)] + \int_0^T E[|b_N(t)|] dt + E[\sup_{0 \leq t \leq T} m_N(t)].$$

However, by the stationarity of u_t , we easily see that

$$E[U_N(0)] = \frac{1}{N} \sum_i E^{\nu_N} [\bar{u}(i)^2 + (\nabla_N \bar{u}(i))^2] \leq C,$$

uniformly in N , where $\bar{u}(i) = u^N(\frac{i}{N}) \stackrel{\text{law}}{=} \nabla_N(\alpha * B(\frac{\cdot}{N}))(i)$, and

$$\begin{aligned} E[|b_N(t)|] &\leq \frac{1}{N} \sum_i (E^{\nu_N} [|\bar{u}(i) \{ \Delta_N \bar{u}(i) + \frac{1}{3} \nabla_N [\alpha_2 * \{\bar{u}(\cdot)^2 + \bar{u}(\cdot - 1)^2 + \bar{u}(\cdot) \bar{u}(\cdot - 1)\}](i) \} \\ &\quad + \nabla_N \bar{u}(i) \{ \nabla_N \Delta_N \bar{u}(i) + \frac{1}{3} \nabla_N^2 [\alpha_2 * \{\bar{u}(\cdot)^2 + \bar{u}(\cdot - 1)^2 + \bar{u}(\cdot) \bar{u}(\cdot - 1)\}](i) \} \\ &\quad + |-N \nabla_N^2 \alpha_2(0) + N \nabla_N^4 \alpha_2(0)|]) \end{aligned}$$

$$\leq C,$$

since $|-N\nabla_N^2\alpha_2(0) + N\nabla_N^4\alpha_2(0)|$ is bounded in N (asymptotically converging to $|\eta_2^\varepsilon(0) + (\eta_2^\varepsilon)'''(0)|$ as $N \rightarrow \infty$), and $E^{\nu_N}[|\nabla_N^\ell \bar{u}(i)|^p]$, $\ell = 0, 1, 2, 3, p \geq 1$ are all independent of i (because of the shift invariance of ν_N) and uniformly bounded in N . Moreover, by Doob's inequality and then by the stationarity of u_t ,

$$\begin{aligned} E[\sup_{0 \leq t \leq T} m_N(t)]^2 &\leq E[\sup_{0 \leq t \leq T} m_N(t)^2] \leq 4E[m_N(T)^2] \\ &= 4 \int_0^T dt \frac{4}{N} \sum_j E[\{\sum_i (u_t(i) \nabla_N \alpha(i-j) + \nabla_N u_t(i) \nabla_N^2 \alpha(i-j))\}^2] \\ &\leq CT. \end{aligned}$$

Note that $\nabla_N \alpha(i-j) = \eta^\varepsilon(\frac{i+1-j}{N}) - \eta^\varepsilon(\frac{i-j}{N})$ and $\nabla_N^2 \alpha(i-j) = N\{\eta^\varepsilon(\frac{i+2-j}{N}) - 2\eta^\varepsilon(\frac{i+1-j}{N}) + \eta^\varepsilon(\frac{i-j}{N})\}$ are both $O(1/N)$. This proves (1).

To show (2), from the definition (2.10) of the linear interpolation for $u_t^N(x)$, one can rewrite $\langle u_t^N, \varphi \rangle_{\mathbb{S}}$ as a sum in i . Then, applying the summation by parts in i , we obtain that

$$(2.13) \quad \langle u_t^N, \varphi \rangle_{\mathbb{S}} = \frac{1}{N} \sum_i u_t(i) \tilde{\varphi}^N(i),$$

where

$$\tilde{\varphi}^N(i) = N^2 \int_{\mathbb{S}} \{1_{[\frac{i-1}{N}, \frac{i}{N})}(x)(x - \frac{i-1}{N}) + 1_{[\frac{i}{N}, \frac{i+1}{N})}(x)(\frac{i+1}{N} - x)\} \varphi(x) dx.$$

However, the Taylor expansion of $\varphi(x)$ around $x = i/N$ in the right hand side up to the third order leads to

$$\tilde{\varphi}^N(i) = \varphi(\frac{i}{N}) + \frac{1}{12N^2} \varphi''(\frac{i}{N}) + r^N(i), \quad i \in \mathbb{T}_N,$$

with remainder terms $r^N(i)$ satisfying $|r^N(i)| \leq C/N^3$. This implies that $|R^N(i)|, |\nabla_N R^N(i)|, |\Delta_N R^N(i)| \leq C/N$ for $R^N(i) := \tilde{\varphi}^N(i) - \varphi(\frac{i}{N})$. Then, from (2.11) and (2.13), for $0 \leq s \leq t$, we have that

$$(2.14) \quad \langle u_t^N - u_s^N, \varphi \rangle_{\mathbb{S}} = I^{(1)} + I^{(2)} + I^{(3)},$$

where

$$\begin{aligned} I^{(1)} &= \frac{1}{2N} \int_s^t \sum_i u_r(i) \{(\Delta_N \varphi(\frac{\cdot}{N}))(i) + \Delta_N R^N(i)\} dr, \\ I^{(2)} &= -\frac{1}{6N} \int_s^t \sum_i \alpha_2 * \{u_r(\cdot)^2 + u_r(\cdot - 1)^2 + u_r(\cdot) u_r(\cdot - 1)\}(i) \\ &\quad \times \{(\nabla_N \varphi(\frac{\cdot}{N}))(i) + \nabla_N R^N(i)\} dr, \\ I^{(3)} &= \frac{1}{\sqrt{N}} \sum_i \nabla_N (w_t^\alpha - w_s^\alpha)(i) \{\varphi(\frac{i}{N}) + R^N(i)\}. \end{aligned}$$

Noting that $E[|u_r(i)|^p] = E^{\nu_N}[|\bar{u}(i)|^p]$ (by stationarity) are bounded in N for $p \geq 1$, we easily see that $E[(I^{(1)})^4], E[(I^{(2)})^4] \leq C(\varphi)(t-s)^4$. Moreover, it is also easy to see that $E[(I^{(3)})^4] \leq C(\varphi)(t-s)^2$. This proves (2). \square

2.4 The martingale problems associated with the SPDEs (2.3) and (2.4)

To complete the proofs of Proposition 2.4 and then Theorem 2.1, we introduce the martingale formulations for the SPDEs (2.3) and (2.4). To this end, we first introduce the (pre) generators of the processes $h(t)$ determined by (1.9) or (1.12).

Let $\mathcal{D} = \mathcal{D}(\mathcal{C})$ be the class of all tame functions Φ on $\mathcal{C} = C(\mathbb{R})$, that is, those of the form:

$$(2.15) \quad \Phi(h) = f(\langle h, \varphi_1 \rangle, \dots, \langle h, \varphi_n \rangle), \quad h \in \mathcal{C},$$

with $n = 1, 2, \dots$, $f = f(z_1, \dots, z_n) \in C_b^2(\mathbb{R}^n)$, $\varphi_1, \dots, \varphi_n \in C_0^\infty(\mathbb{R})$, where $\langle h, \varphi \rangle = \int_{\mathbb{R}} h(x) \varphi(x) dx$. We define its functional derivatives by

$$(2.16) \quad D\Phi(x; h) = \sum_{i=1}^n \partial_{z_i} f(\langle h, \varphi_1 \rangle, \dots, \langle h, \varphi_n \rangle) \varphi_i(x),$$

$$(2.17) \quad D^2\Phi(x_1, x_2; h) = \sum_{i,j=1}^n \partial_{z_i} \partial_{z_j} f(\langle h, \varphi_1 \rangle, \dots, \langle h, \varphi_n \rangle) \varphi_i(x_1) \varphi_j(x_2).$$

The class $\mathcal{D}_\infty = \mathcal{D}_\infty(\mathcal{C})$ stands for the family of all $\Phi \in \mathcal{D}$ determined by (2.15) with $f \in C_\infty^2(\mathbb{R}^n)$ such that $\lim_{|z| \rightarrow \infty} \{|f(z)| + |\partial_{z_i} f(z)| + |\partial_{z_i} \partial_{z_j} f(z)|\} = 0$.

For $\Phi \in \mathcal{D}$, define two operators $\mathcal{L}_0^\varepsilon$ and \mathcal{A}^ε by

$$\begin{aligned} \mathcal{L}_0^\varepsilon \Phi(h) &= \frac{1}{2} \int_{\mathbb{R}^2} D^2\Phi(x_1, x_2; h) \eta_2^\varepsilon(x_1 - x_2) dx_1 dx_2 + \frac{1}{2} \int_{\mathbb{R}} \partial_x^2 h(x) D\Phi(x; h) dx, \\ \mathcal{A}^\varepsilon \Phi(h) &= \frac{1}{2} \int_{\mathbb{R}} ((\partial_x h)^2 - \xi^\varepsilon) * \eta_2^\varepsilon(x) D\Phi(x; h) dx. \end{aligned}$$

Then, $\mathcal{L}^\varepsilon := \mathcal{L}_0^\varepsilon + \mathcal{A}^\varepsilon$ is the (formal) generator corresponding to the SPDE (1.9). In fact, by applying Itô's formula, we have that

$$d\Phi(h_t) = \langle D\Phi(x; h_t), dh_t(x) \rangle_{\mathbb{R}} + \frac{1}{2} \langle D^2\Phi(x_1, x_2; h_t), dW^\varepsilon(t, x_1) dW^\varepsilon(t, x_2) \rangle_{\mathbb{R}^2}$$

and note that

$$(2.18) \quad dW^\varepsilon(t, x_1) dW^\varepsilon(t, x_2) = \eta_2^\varepsilon(x_1 - x_2) dt.$$

The (formal) generator corresponding to the SPDE (2.3) for the tilt process $u = \partial_x h$ is given by $\mathcal{L}^{\varepsilon, U} = \mathcal{L}_0^{\varepsilon, U} + \mathcal{A}^{\varepsilon, U}$, where

$$\begin{aligned} \mathcal{L}_0^{\varepsilon, U} \Phi(u) &= \frac{1}{2} \int_{\mathbb{R}^2} D^2\Phi(x_1, x_2; u) \partial_{x_1} \partial_{x_2} \{\eta_2^\varepsilon(x_1 - x_2)\} dx_1 dx_2 + \frac{1}{2} \int_{\mathbb{R}} \partial_x^2 u(x) D\Phi(x; u) dx, \\ \mathcal{A}^{\varepsilon, U} \Phi(u) &= \frac{1}{2} \int_{\mathbb{R}} \partial_x (u^2 * \eta_2^\varepsilon)(x) D\Phi(x; u) dx, \end{aligned}$$

for $\Phi = \Phi(u) \in \mathcal{D}$, which is given by (2.15) with u in place of h . Note that the derivatives ∂_x^2 and ∂_x in these operators can be moved to $D\Phi(x; u)$ by the integration by parts.

We similarly define $\mathcal{D}(\mathcal{C}_M)$ and $\mathcal{D}(\mathcal{C}_{M,0})$ as the classes of all Φ on \mathcal{C}_M and $\mathcal{C}_{M,0}$, respectively, of the forms (2.15) with $\varphi_i \in C^\infty(\mathbb{S}_M)$ and $\langle h, \varphi \rangle_{\mathbb{S}_M} := \int_{\mathbb{S}_M} h(x) \varphi(x) dx$ in place of $\langle h, \varphi \rangle$. Then, operators $\mathcal{L}_{0,M}^\varepsilon, \mathcal{A}_M^\varepsilon$ together with $\mathcal{L}_M^\varepsilon := \mathcal{L}_{0,M}^\varepsilon + \mathcal{A}_M^\varepsilon$ on $\mathcal{D}(\mathcal{C}_M)$ and $\mathcal{L}_{0,M}^{\varepsilon,U}, \mathcal{A}_M^{\varepsilon,U}$ together with $\mathcal{L}_M^{\varepsilon,U} := \mathcal{L}_{0,M}^{\varepsilon,U} + \mathcal{A}_M^{\varepsilon,U}$ on $\mathcal{D}(\mathcal{C}_{M,0})$ are defined as $\mathcal{L}_0^\varepsilon, \mathcal{A}^\varepsilon, \mathcal{L}^\varepsilon$ and $\mathcal{L}_{0,M}^{\varepsilon,U}, \mathcal{A}_M^{\varepsilon,U}, \mathcal{L}_M^{\varepsilon,U}$, respectively, by replacing the integrals over \mathbb{R}^2 and \mathbb{R} by those over \mathbb{S}_M^2 and \mathbb{S}_M , respectively. We also consider the classes of functions $\mathcal{D}_\infty(\mathcal{C}_M)$ and $\mathcal{D}_\infty(\mathcal{C}_{M,0})$.

Remark 2.1. We can regard \mathcal{L}^ε as the generator of the tilt process u by replacing its domain. In fact, let $\mathcal{D}_\nabla = \mathcal{D}_\nabla(\mathcal{C})$ be the class of all $\Phi \in \mathcal{D}$ with φ_i satisfying $\int_{\mathbb{R}} \varphi_i dx = 0, 1 \leq i \leq n$. This is a natural class of functions for tilt variables, since, under the equivalence relation $h \sim h + c$ with some $c \in \mathbb{R}$, we have $\Phi(h) = \Phi(h + c)$ if $\Phi \in \mathcal{D}_\nabla$ so that Φ is a function on the quotient space $\tilde{\mathcal{C}} = \mathcal{C}/\sim$. For the function $\Phi \in \mathcal{D}_\nabla$, though we write its variable by h , the height h itself has no meaning. In particular, if h is differentiable, $\Phi \in \mathcal{D}_\nabla$ can be considered as a function of its tilt $u := h' \equiv \partial_x h$: if $\Phi(u) = f(\langle u, \psi_1 \rangle, \dots, \langle u, \psi_n \rangle)$ with $\psi_1, \dots, \psi_n \in C_0^\infty(\mathbb{R})$, then $\langle u, \psi_i \rangle = \langle h, \varphi_i \rangle$ with $\varphi_i := -\psi_i'$ and φ_i satisfies the condition $\int_{\mathbb{R}} \varphi_i dx = 0$, which is the additional condition imposed on $\Phi \in \mathcal{D}_\nabla$. We can also define $\mathcal{D}_\nabla(\mathcal{C}_M)$ as the class of all $\Phi \in \mathcal{D}(\mathcal{C}_M)$ with $\varphi_i \in C^\infty(\mathbb{S}_M)$ satisfying $\int_{\mathbb{S}_M} \varphi_i(x) dx = 0$.

We now introduce the martingale problems associated with the SPDEs (2.3) and (2.4) on extended spaces. Recall that $\varepsilon > 0$ is fixed so that the noise $\dot{W}^\varepsilon(t, x)$ is smooth in x . As a state spaces for the SPDE (2.3), we take $C(\mathbb{R}) \cap L_r^2(\mathbb{R}), r > 0$, where $L_r^2(\mathbb{R})$ is the weighted L^2 -space; recall (1.15).

Lemma 2.7. (1) If the probability measure P on $C([0, \infty), C(\mathbb{R}) \cap L_r^2(\mathbb{R}))$ is a solution of the $(\mathcal{L}^{\varepsilon,U}, \mathcal{D}_\infty)$ -martingale problem, then there exists $\dot{W}^\varepsilon(t, x)$, which is defined on this space and a Gaussian smeared noise under P , such that the coordinate function $u(t)$ is a solution of the SPDE (2.3) in the generalized functions' sense; i.e., (2.3) holds multiplied by any test function $\varphi \in C_0^\infty(\mathbb{R})$ and integrated over \mathbb{R} (as in (1.4)).

(2) Similar results hold on \mathbb{S}_M : Under the solution P on $C([0, \infty), \mathcal{C}_{M,0})$ of the $(\mathcal{L}_M^{\varepsilon,U}, \mathcal{D}_\infty(\mathcal{C}_{M,0}))$ -martingale problem, the coordinate function $u(t)$ satisfies the SPDE (2.4) in the generalized functions' sense with a certain Gaussian smeared noise $\dot{W}^\varepsilon(t, x)$ on $[0, \infty) \times \mathbb{S}_M$.

Proof. To prove (1), we use two types of functions $\Phi_1(u) = \langle u, \varphi \rangle$ and $\Phi_2(u) = \langle u, \varphi_1 \rangle \langle u, \varphi_2 \rangle$; more precisely, their cut-off functions such as $\Phi_{1,N}(u) = g_N(\langle u, \varphi \rangle)$ with $g_N \in C_\infty^2(\mathbb{R})$ satisfying $g_N(x) = x$ for $|x| \leq N$ and similarly defined functions $\Phi_{2,N}$ for Φ_2 . We denote by $b(u, \varphi) = \frac{1}{2} \langle \partial_x^2 u + \partial_x(u^2 * \eta_2^\varepsilon), \varphi \rangle$. Then,

$$(2.19) \quad \begin{aligned} M_t(\varphi) &:= \langle u(t), \varphi \rangle - \langle u(0), \varphi \rangle - \int_0^t \mathcal{L}^{\varepsilon,U} \Phi_1(u(s)) ds \\ &= \langle u(t), \varphi \rangle - \langle u(0), \varphi \rangle - \int_0^t b(u(s), \varphi) ds \end{aligned}$$

is a local martingale. Moreover, by noting that $\Phi_2(u(t)) - \int_0^t \mathcal{L}^{\varepsilon,U} \Phi_2(u(s)) ds$ is a local martingale and $\mathcal{L}^{\varepsilon,U} \Phi_2(u) = \langle \partial_{x_1} \partial_{x_2} \{ \eta_2^\varepsilon(-\cdot) \}, \varphi_1 \otimes \varphi_2 \rangle_{\mathbb{R}^2} + b(u, \varphi_1) \langle u, \varphi_2 \rangle + b(u, \varphi_2) \langle u, \varphi_1 \rangle$, and by applying Itô's formula, we can easily see that

$$(2.20) \quad M_t(\varphi_1) M_t(\varphi_2) - t \int_{\mathbb{R}^2} \partial_{x_1} \partial_{x_2} \{ \eta_2^\varepsilon(x_1 - x_2) \} \varphi_1(x_1) \varphi_2(x_2) dx_1 dx_2$$

is a local martingale. This implies that the cross variation of two local martingales $M_t(\varphi_1)$ and $M_t(\varphi_2)$ with $\varphi_1, \varphi_2 \in C_0^\infty(\mathbb{R})$ is given by

$$(2.21) \quad \langle M(\varphi_1), M(\varphi_2) \rangle_t = t \langle V\varphi_1, \varphi_2 \rangle,$$

where the right hand side denotes the inner product in $L^2(\mathbb{R}) = L^2(\mathbb{R}, dx)$ and

$$V\varphi(x) = \partial_x \int_{\mathbb{R}} \eta_2^\varepsilon(x - x_1) (-\partial_{x_1}) \varphi(x_1) dx_1.$$

Introducing operators: $R = \partial_x$ and

$$\begin{aligned} Q\varphi(x) &= \int_{\mathbb{R}} \eta_2^\varepsilon(x - x_1) \varphi(x_1) dx_1, \\ Q^{\frac{1}{2}}\varphi(x) &= \int_{\mathbb{R}} \eta^\varepsilon(x - x_1) \varphi(x_1) dx_1, \end{aligned}$$

we can rewrite V as $V = (RQ^{\frac{1}{2}})(RQ^{\frac{1}{2}})^*$ as operators on $L^2(\mathbb{R})$. Note that Q and $Q^{\frac{1}{2}}$ are symmetric on $L^2(\mathbb{R})$, $(Q^{\frac{1}{2}})^2 = Q$, and, in particular, Q is non-negative, but $\|Q^{\frac{1}{2}}\|_{\text{HS}}^2 = \int_{\mathbb{R}^2} (\eta^\varepsilon(x_1 - x_2))^2 dx_1 dx_2 = \infty$ so that $\text{Tr } Q = \infty$.

By the martingale representation theorem (see, e.g., [6] Theorem 8.2, actually stated only in case $\text{Tr } Q < \infty$, and also Remark 2.2 below), (2.21) implies that

$$M_t(x) = RW^Q(t, x) \equiv \partial_x W^Q(t, x),$$

where W^Q is the Q -Wiener process, which has the representation (2.1) with a space-time Gaussian white noise. This with (2.19) implies the conclusion of (1). The proof of (2) is similar. \square

Remark 2.2. *In our case, as we pointed out, the assumption $\text{Tr } Q < \infty$ is not satisfied. To overcome this, we may first define $M_t^N(x)$ by restricting $M_t(x)$ on $[-N, N]$ and periodically extending it to $[-N - 2\varepsilon, N + 2\varepsilon]$. For M_t^N , the corresponding Q -operator is given by*

$$Q^N\varphi(x) = \int_{-N-2\varepsilon}^{N+2\varepsilon} \eta_2^\varepsilon(x - x_1) \varphi(x_1) dx_1, \quad x \in [-N, N],$$

with φ defined on $[-N, N]$ but periodically extended to $[-N - 2\varepsilon, N + 2\varepsilon]$ and this operator becomes of trace class. Therefore, one can apply Theorem 8.2 of [6] and construct $W^{Q^N}(t, x)$. Then, by the consistency of $W^{Q^N}(t, x)$, one can extend it to the whole line \mathbb{R} .

2.5 Proof of Proposition 2.4

We may assume $M = 1$ without loss of generality. Recall that $\{u_t^N(t); x \in \mathbb{S}\}$ is defined by the linear interpolation of the stationary solution $u_t = \{u_t(i)\}$ of (2.11) in such a manner that $u_t^N(\frac{i}{N}) = u_t(i)$, $i \in \mathbb{T}_N$, and it is tight on $C([0, T], C(\mathbb{S}))$ from Lemma 2.6. Therefore, by Skorohod's representation theorem, we can realize on a proper probability space such that u_t^N converges to some u_t in $C([0, T], C(\mathbb{S}))$ as $N \rightarrow \infty$ a.s. for every $T > 0$. We abuse the notation. Then, for every $\Phi \in \mathcal{D}(C(\mathbb{S}))$, we have that

$$(2.22) \quad d\Phi(u_t^N) = \langle D\Phi(\cdot; u_t^N), du_t^N \rangle_{\mathbb{S}} + \frac{1}{2} \langle D^2\Phi(x_1, x_2; u_t^N), du_t^N(x_1) du_t^N(x_2) \rangle_{\mathbb{S}^2}.$$

However, from (2.14), we have that

$$\begin{aligned}
(2.23) \quad d\langle u_t^N, \varphi \rangle &= \frac{1}{2N} \sum_i u_t^N(\frac{i}{N}) \{(\Delta_N \varphi(\frac{\cdot}{N}))(i) + \Delta_N R^N(i)\} dt \\
&\quad - \frac{1}{6N} \sum_i \alpha_2 * \{u_t(\frac{\cdot}{N})^2 + u_t(\frac{\cdot-1}{N})^2 + u_t(\frac{\cdot}{N})u_t(\frac{\cdot-1}{N})\}(i) \\
&\quad \quad \times \{(\nabla_N \varphi(\frac{\cdot}{N}))(i) + \nabla_N R^N(i)\} dt \\
&\quad - \frac{1}{\sqrt{N}} \sum_i \{(\nabla_N \varphi(\frac{\cdot}{N}))(i) + \nabla_N R^N(i)\} \sum_j \alpha(i-j) dw_t(j).
\end{aligned}$$

In particular, recalling that $|\nabla_N R^N(i)| \leq C/N$,

$$\begin{aligned}
&d\langle u_t^N, \varphi_1 \rangle d\langle u_t^N, \varphi_2 \rangle \\
&= \frac{1}{N} \sum_{i,j} \alpha_2(i-j) \{(\nabla_N \varphi_1(\frac{\cdot}{N}))(i) + \nabla_N R_1^N(i)\} \{(\nabla_N \varphi_2(\frac{\cdot}{N}))(j) + \nabla_N R_2^N(j)\} dt \\
&= \frac{1}{N^3} \sum_{i,j,k} \eta^\varepsilon(\frac{i-k}{N}) \eta^\varepsilon(\frac{k-j}{N}) \{(\nabla_N \varphi_1(\frac{\cdot}{N}))(i) + \nabla_N R_1^N(i)\} \{(\nabla_N \varphi_2(\frac{\cdot}{N}))(j) + \nabla_N R_2^N(j)\} dt \\
&\rightarrow \int_{\mathbb{S}^3} \eta^\varepsilon(x-z) \eta^\varepsilon(y-z) \varphi_1'(x) \varphi_2'(y) dx dy dz \cdot dt = \int_{\mathbb{S}^2} \eta_2^\varepsilon(x-y) \varphi_1'(x) \varphi_2'(y) dx dy \cdot dt,
\end{aligned}$$

as $N \rightarrow \infty$. Since u_t^N converges to u_t in the space $C([0, T], C(\mathbb{S}))$ a.s., for the limit u_t , we see from (2.22) and (2.23) that

$$\Phi(u_t) - \int_0^t \mathcal{L}_1^{\varepsilon, U} \Phi(u_s) ds$$

is a martingale for every $\Phi \in \mathcal{D}(C(\mathbb{S}))$ and therefore for $\Phi \in \mathcal{D}(\mathcal{C}_{1,0})$. This completes the proof of Proposition 2.4 with the help of Lemma 2.7-(2) and Lemma 2.5.

2.6 Invariant measure of KPZ approximating equation on \mathbb{R}

Let $u_t^M = \{u_t^M(x); x \in \mathbb{S}_M = [0, M)\}$ be the stationary solution of the SPDE (2.4), that is, $u_0^M \stackrel{\text{law}}{=} \nu^{\varepsilon, M}$, constructed in Proposition 2.4. We extend u_t^M periodically on \mathbb{R} .

Lemma 2.8. (1) *For every $T > 0$ and $r > 0$, we have*

$$\sup_{M \geq 1} E \left[\sup_{0 \leq t \leq T} \|u_t^M\|_{H_r^1(\mathbb{R})}^2 \right] < \infty,$$

where $\|u\|_{H_r^1(\mathbb{R})}^2 = \|u\|_{L_r^2(\mathbb{R})}^2 + \|\partial_x u\|_{L_r^2(\mathbb{R})}^2$.

(2) *For every $T > 0$, $\varphi \in C_0^\infty(\mathbb{R})$ and $0 \leq s < t \leq T$,*

$$E[\langle u_t^M - u_s^M, \varphi \rangle^4] \leq C(\varphi)(t-s)^2,$$

holds with $C(\varphi) = C_T(\varphi) > 0$.

(3) *In particular, $\{u_t^M\}_{M \geq 1}$ is tight on $C([0, T], C(\mathbb{R}) \cap L_r^2(\mathbb{R}))$ for every $T, r > 0$.*

Proof. The proof is parallel to that of Lemma 2.6. Indeed, (3) follows from (1) and (2) noting that the embedding $H_r^1(\mathbb{R}) \subset H_{r'}^s(\mathbb{R})$ is compact if $r' > r > 0$ and $s < 1$, and also $H_r^s(\mathbb{R}) \subset C(\mathbb{R})$ if $s > 1/2$; see [10], p.284 for the weighted Sobolev spaces $H_r^s(\mathbb{R})$.

To show (1), set $U^M(t) = \|u_t^M\|_{H_r^1(\mathbb{R})}^2$. Then, by Itô's formula,

$$\begin{aligned} dU^M(t) &= \int_{\mathbb{R}} \{2u_t(x)du_t(x) + (du_t(x))^2 + 2\partial_x u_t(x)d\partial_x u_t(x) + (d\partial_x u_t(x))^2\} e^{-2r\chi(x)} dx \\ &= \int_{\mathbb{R}} [u_t(x)\{\partial_x^2 u_t(x) + \partial_x(u_t^2 * \eta_2^\varepsilon)\} + \partial_x u_t(x)\{\partial_x^3 u_t(x) + \partial_x^2(u_t^2 * \eta_2^\varepsilon)\} \\ &\quad + (-(\eta_2^\varepsilon)''(0) + (\eta_2^\varepsilon)''''(0))] e^{-2r\chi(x)} dx \cdot dt \\ &\quad + \int_{\mathbb{R}} \{2u_t(x)d\partial_x W^\varepsilon(t, x) + 2\partial_x u_t(x)d\partial_x^2 W^\varepsilon(t, x)\} e^{-2r\chi(x)} dx \\ &=: b^M(t)dt + dm^M(t), \end{aligned}$$

where $u_t = u_t^M$ and $W^\varepsilon(t, x)$ originally defined on \mathbb{S}_M is periodically extended on \mathbb{R} . We can bound $E[\sup_{0 \leq t \leq T} U^M(t)]$ by the sum of three terms similarly to (2.12). However, we easily see that

$$E[U^M(0)] = E^{\nu^{\varepsilon, M}}[\|\partial_x B^M * \eta^\varepsilon\|_{H_r^1(\mathbb{R})}^2] \leq C \quad (\text{uniformly in } M),$$

where $\{B^M(x); x \in \mathbb{S}_M\}$ is periodically extended on \mathbb{R} , and

$$\begin{aligned} E[|b^M(t)|] &\leq \int_{\mathbb{R}} e^{-2r\chi(x)} dx (E^{\nu^{\varepsilon, M}}[|u(x)\{\partial_x^2 u(x) + \partial_x(u^2 * \eta_2^\varepsilon)\} \\ &\quad + \partial_x u(x)\{\partial_x^3 u(x) + \partial_x^2(u^2 * \eta_2^\varepsilon)\}|] + (-(\eta_2^\varepsilon)''(0) + (\eta_2^\varepsilon)''''(0))) \\ &\leq C, \end{aligned}$$

since $E^{\nu^{\varepsilon, M}}[|\partial_x^\ell u(x)|^p]$, $\ell = 0, 1, 2, 3, p \geq 1$ are all independent of x (because of the shift invariance of $u(x)$ under $\nu^{\varepsilon, M}$) and uniformly bounded in M . Moreover, by Doob's inequality and then by the stationarity of u_t ,

$$\begin{aligned} E[\sup_{0 \leq t \leq T} m^M(t)]^2 &\leq 4 \int_0^T dt \int_{\mathbb{R}^2} 8e^{-2r(\chi(x)+\chi(y))} dx dy \{E^{\nu^{\varepsilon, M}}[u(x)u(y)]\partial_x \partial_y \eta_{2, M}^\varepsilon(x-y) \\ &\quad + E^{\nu^{\varepsilon, M}}[\partial_x u(x)\partial_y u(y)]\partial_x^2 \partial_y^2 \eta_{2, M}^\varepsilon(x-y)\} \\ &\leq CT, \end{aligned}$$

where $\eta_{2, M}^\varepsilon(x-y)$ is defined in the sense of modulo M in $x-y$; note that $\partial_x \partial_y \eta_{2, M}^\varepsilon$ and $\partial_x^2 \partial_y^2 \eta_{2, M}^\varepsilon$ are bounded in M, x, y . We have estimated as $m^M(T)^2 \leq 2(m_1^M(T)^2 + m_2^M(T)^2)$ by decomposing $m^M(T)$ into the sum of two stochastic integrals $m_1^M(T)$ and $m_2^M(T)$. This proves (1).

For (2), denoting $u_t = u_t^M$ again, we see that

$$\begin{aligned} \langle u_t - u_s, \varphi \rangle &= \frac{1}{2} \int_s^t \{\langle u_r, \varphi'' \rangle - \langle u_r^2 * \eta_2^\varepsilon, \varphi' \rangle\} dr - \{W^\varepsilon(t, \varphi') - W^\varepsilon(s, \varphi')\} \\ &=: I^{(1)} + I^{(2)}. \end{aligned}$$

However, we easily see that $E[(I^{(1)})^4] \leq C(\varphi)(t-s)^4$ by the stationarity of u_t , and $E[(I^{(2)})^4] = C(\varphi)(t-s)^2$, since $W^\varepsilon(t, \varphi')$ is a Brownian motion multiplied by a certain constant. This proves (2). \square

Let $u = \{u(x); x \in \mathbb{S}_M\}$ be a $\mathcal{C}_{M,0}$ -valued random variable distributed under $\nu^{\varepsilon, M}$ and, by periodically extending u on \mathbb{R} , we can regard $\nu^{\varepsilon, M}$ as a probability distribution on \mathcal{C} . Then, the following lemma is easy and the proof is omitted.

Lemma 2.9. *The distribution $\nu^{\varepsilon, M}$ weakly converges to ν^ε on the space \mathcal{C} as $M \rightarrow \infty$.*

We are now ready to give the proof of Theorem 2.1.

Proof of Theorem 2.1. The assertion (1) is already shown by Proposition 2.4. Let us prove (2). We have shown in Lemma 2.8 that the periodically extended stationary solution $\{u_t^M\}_{M \geq 1}$ of the SPDE (2.4) is tight on $C([0, T], C(\mathbb{R}) \cap L_r^2(\mathbb{R}))$ for every $r > 0$. Therefore, by Skorohod's representation theorem, we can realize on a proper probability space that u_t^M converges to some u_t in $C([0, T], C(\mathbb{R}) \cap L_r^2(\mathbb{R}))$ for every $T, r > 0$ as $M \rightarrow \infty$ a.s. Then, for every $\Phi \in \mathcal{D}(\mathcal{C})$,

$$\Phi(u_t^M) - \int_0^t \mathcal{L}_M^{\varepsilon, U} \Phi(u_s^M) ds$$

is a martingale. Here, in the operator $\mathcal{L}_M^{\varepsilon, U}$, the function η_2^ε should be understood in the sense of modulo M . However, noting that the supports of the functions $\varphi_1, \dots, \varphi_n$ appearing in Φ are compact, we see that $\mathcal{L}_M^{\varepsilon, U} \Phi(u^M)$ converges to $\mathcal{L}^{\varepsilon, U} \Phi(u)$ as $M \rightarrow \infty$ if u^M converges to u in $C([0, T], C(\mathbb{R}) \cap L_r^2(\mathbb{R}))$. Thus, one can prove that, for the limit u_t ,

$$(2.24) \quad \Phi(u_t) - \int_0^t \mathcal{L}^{\varepsilon, U} \Phi(u_s) ds$$

is a martingale for every $\Phi \in \mathcal{D}(\mathcal{C})$. This completes the proof of Theorem 2.1-(2) with the help of Lemma 2.7-(1) and Lemma 2.9. \square

As a corollary, we can prove the infinitesimal invariance of $\mathcal{L}^{\varepsilon, U}$, the symmetry of $\mathcal{L}_0^{\varepsilon, U}$ and the asymmetry of $\mathcal{A}^{\varepsilon, U}$ under ν^ε , respectively, or integration by parts formulas, and similar results on \mathbb{S}_M .

Corollary 2.10. (1) *For every $\varepsilon > 0$ and $\Phi \in \mathcal{D}(\mathcal{C})$, we have that*

$$(2.25) \quad \int \mathcal{L}^{\varepsilon, U} \Phi d\nu^\varepsilon = 0.$$

The operators $\mathcal{L}_0^{\varepsilon, U}$ and $\mathcal{A}^{\varepsilon, U}$ are symmetric and asymmetric with respect to ν^ε , respectively, that is, for every $\Phi, \Psi \in \mathcal{D}(\mathcal{C})$,

$$(2.26) \quad \int \Psi \mathcal{L}_0^{\varepsilon, U} \Phi d\nu^\varepsilon = \int \Phi \mathcal{L}_0^{\varepsilon, U} \Psi d\nu^\varepsilon,$$

and

$$(2.27) \quad \int \Psi \mathcal{A}^{\varepsilon, U} \Phi d\nu^\varepsilon = - \int \Phi \mathcal{A}^{\varepsilon, U} \Psi d\nu^\varepsilon.$$

(2) Similar results hold on \mathbb{S}_M with $\mathcal{L}_M^{\varepsilon,U}$, $\mathcal{L}_{0,M}^{\varepsilon,U}$, $\mathcal{A}_M^{\varepsilon,U}$ and $\nu^{\varepsilon,M}$ in place of $\mathcal{L}^{\varepsilon,U}$, $\mathcal{L}_0^{\varepsilon,U}$, $\mathcal{A}^{\varepsilon,U}$ and ν^ε , respectively.

Proof. We give the proof of (1) only. (2.25) follows by taking the average of the martingale (2.24) and noting that $u_t \stackrel{\text{law}}{=} \nu^\varepsilon$. (2.26) can be shown from (2.6) rewritten at the tilt level and by taking the limits twice as we did, or it can be directly shown by noting that ν^ε is reversible for the Ornstein-Uhlenbeck process determined by the SPDE:

$$\partial_t u = \frac{1}{2} \partial_x^2 u + \partial_x \dot{W}^\varepsilon(t, x), \quad x \in \mathbb{R}.$$

Since $\mathcal{A}^{\varepsilon,U} = \mathcal{L}^{\varepsilon,U} - \mathcal{L}_0^{\varepsilon,U}$, (2.25) and (2.26) with $\Psi = 1$ prove that

$$(2.28) \quad \int \mathcal{A}^{\varepsilon,U} \Phi d\nu^\varepsilon = 0,$$

and (2.27) follows from this by noting that $\mathcal{A}^{\varepsilon,U}(\Phi\Psi) = \Psi\mathcal{A}^{\varepsilon,U}\Phi + \Phi\mathcal{A}^{\varepsilon,U}\Psi$. \square

Remark 2.3. We can alternatively prove the infinitesimal invariance (2.25) directly using the Wiener-Itô expansion, see [11].

Remark 2.4. (Yaglom reversibility) Corollary 2.10 suggests that the generator of the time reversed process under ν^ε is given by $\mathcal{L}_0^{\varepsilon,U} - \mathcal{A}^{\varepsilon,U}$. Coming back to the level of the height processes, a simple computation shows that $\mathcal{L}_0^{\varepsilon}\check{\Phi}(h) = \mathcal{L}_0^{\varepsilon}\Phi(\check{h})$ and $\mathcal{A}^{\varepsilon}\check{\Phi}(h) = -\mathcal{A}^{\varepsilon}\Phi(\check{h})$ for $\Phi \in \mathcal{D}$, where \check{h} and $\check{\Phi}$ are defined by the transformations $\check{h}(x) = -h(-x)$ and $\check{\Phi}(h) = \Phi(\check{h})$, respectively. This means that $\check{h}(t, x) := -h(t, -x)$ determined from the solution $h(t, x)$ of the SPDE (1.9) admits the (pre) generator $\mathcal{L}_0^{\varepsilon} - \mathcal{A}^{\varepsilon}$.

2.7 Invariant measure for the height process

Theorem 2.1 deals with the tilt processes only, but this can be easily extended to the height process. Theorem 2.11 will not be used later, but we state it for its own interest. Set

$$X_t^\varepsilon = \frac{1}{2} \int_0^t \partial_x^2 h^\varepsilon(s, 0) ds + \frac{1}{2} \int_0^t ((\partial_x h^\varepsilon(s))^2 - \xi^\varepsilon) * \eta_2^\varepsilon(0) ds + W^\varepsilon(t, 0),$$

for the solution $h^\varepsilon(t, x)$ of (1.9). The key point is that, as functions of h^ε , the first and second terms of X_t^ε are defined on the quotient space $\tilde{\mathcal{C}}$ defined in Remark 2.1. Therefore, once $h^\varepsilon(t) \in \tilde{\mathcal{C}}$ is determined by solving the SPDE (2.3), we can recover its height at $x = 0$ as

$$(2.29) \quad h^\varepsilon(t, 0) = h^\varepsilon(0, 0) + X_t^\varepsilon.$$

Theorem 2.11. For any bounded, integrable and continuous function $G = G(h_0, h)$ on $\mathbb{R} \times \tilde{\mathcal{C}}$ and for any $\varepsilon > 0$, $t \geq 0$, we have that

$$(2.30) \quad \int_{\mathbb{R} \times \tilde{\mathcal{C}}} G(h^\varepsilon(t, 0), h^\varepsilon(t)) dh_0 d\nu^\varepsilon = \int_{\mathbb{R} \times \tilde{\mathcal{C}}} G(h^\varepsilon(0, 0), h^\varepsilon(0)) dh_0 d\nu^\varepsilon,$$

where dh_0 means that $h^\varepsilon(0, 0)$ is distributed under the Lebesgue measure on \mathbb{R} . Note that $\mathbb{R} \times \tilde{\mathcal{C}}$ can be identified with \mathcal{C} . Similar results hold on \mathbb{S}_M .

Proof. From (2.29) and then by the translation-invariance of the Lebesgue measure and performing the integral in dh_0 first, the left hand side of (2.30) is equal to

$$\int_{\mathbb{R} \times \tilde{\mathcal{C}}} G(h^\varepsilon(0, 0) + X_t^\varepsilon, h^\varepsilon(t)) dh_0 d\nu^\varepsilon = \int_{\mathbb{R} \times \tilde{\mathcal{C}}} G(h^\varepsilon(0, 0), h^\varepsilon(t)) dh_0 d\nu^\varepsilon.$$

But, this is equal to the right hand side of (2.30) by the invariance of ν^ε under $h^\varepsilon(t) \in \tilde{\mathcal{C}}$ due to Theorem 2.1-(2). \square

3 Cole-Hopf transform of KPZ approximating equation and proofs of Theorems 1.1 and 1.2

Our goal is to study the limit of the KPZ approximating equation (1.9) on \mathbb{R} or (1.12) on \mathbb{S}_M as $\varepsilon \downarrow 0$. To this end, we move to the level of the corresponding Cole-Hopf transformed process $Z^\varepsilon(t)$ rather than staying with (1.9) or (1.12), and show that $Z^\varepsilon(t)$ converges to the solution $Z(t)$ of the SPDE (1.11) on \mathbb{R} or (1.14) on \mathbb{S}_M at least if the corresponding tilt process is stationary. This implies that the solution $h^\varepsilon(t)$ of (1.9) or (1.12) converges to $h(t) + \frac{1}{24}t$ as $\varepsilon \downarrow 0$, where $h(t)$ is the Cole-Hopf solution of the KPZ equation defined by (1.6). We can actually do this only for (1.12); due to a technical reason, we do not have Proposition 3.13 on \mathbb{R} . Since all arguments except this do work on \mathbb{R} , we state the results on \mathbb{R} in Sections 3.1–3.3. Then, we study the SPDEs on \mathbb{S}_M in Sections 3.4 and 3.5. Finally in Section 3.6, letting $M \rightarrow \infty$, as a byproduct, we find an invariant measure of the SHE (1.3) on \mathbb{R} .

3.1 The equation for $Z^\varepsilon(t)$

Under the transformation $h \mapsto Z$ defined by $Z = e^h$, the KPZ approximating equation (1.9) is transformed into the equation (1.10) for $Z = Z^\varepsilon(t)$. In fact, by applying Itô's formula and recalling (2.18) with $x_1 = x_2 = x$,

$$\begin{aligned} dZ &= e^h dh + \frac{1}{2} e^h (dW^\varepsilon)^2 \\ &= \frac{1}{2} Z \left(\partial_x^2 h + ((\partial_x h)^2 - \xi^\varepsilon) * \eta_2^\varepsilon \right) dt + Z dW^\varepsilon + \frac{1}{2} Z \xi^\varepsilon dt \\ &= \frac{1}{2} Z \left(\partial_x^2 h + (\partial_x h)^2 * \eta_2^\varepsilon \right) dt + Z dW^\varepsilon. \end{aligned}$$

Thus, (1.10) is obtained noting that $\partial_x^2 h + (\partial_x h)^2 = Z^{-1} \partial_x^2 Z$ and $\partial_x h = \partial_x Z / Z$. The derivation of (1.13) from (1.12) is the same.

We define the notion of tilt variables associated with the process $Z(t)$. This is a reformulation of those defined for h above Remark 1.1 or in Remark 2.1. For $Z^1, Z^2 \in \mathcal{C}_+$, we say $Z^1 \sim Z^2$ if there exists $c > 0$ such that $Z^1(x) = cZ^2(x)$ for all $x \in \mathbb{R}$. Then, by the linearity and uniqueness of solutions of the SPDE (1.3), we see that $Z^1(t) \sim Z^2(t)$ holds if $Z^1(0) \sim Z^2(0)$ for two solutions $Z^1(t), Z^2(t)$ of (1.3). Thus, (1.3) defines a stochastic evolution $\tilde{Z}(t)$ on the quotient space $\tilde{\mathcal{C}}_+ := \mathcal{C}_+ / \sim$. The SPDE (1.10) has the same character, though it is nonlinear.

3.2 Wrapped processes

To avoid the complexity arising from the infiniteness of the invariant measure of $h^\varepsilon(t, x)$, we introduce a modified process $g^\varepsilon(t, x)$ of $h^\varepsilon(t, x)$. Let us take $\rho \in C_0^\infty(\mathbb{R})$ satisfying $\rho \geq 0$, $\text{supp } \rho \subset [-1, 1]$, $\text{supp } \rho$ is connected, and $\int_{\mathbb{R}} \rho(x) dx = 1$, and fix it in the rest of the paper except the last step of Section 3.5, where we take two different ρ 's. We define a wrapped process $g^\varepsilon(t, x)$ of $h^\varepsilon(t, x)$ by $g^\varepsilon(t, x) = h^\varepsilon(t, x) + N^\varepsilon(t)$ with $N^\varepsilon(t) = -[h^\varepsilon(t, \rho)]$, more precisely its right continuous modification, where $[h] \in \mathbb{Z}$ stands for the integer part of $h \in \mathbb{R}$ and $h^\varepsilon(t, \rho) = \int_{\mathbb{R}} h^\varepsilon(t, x) \rho(x) dx$. In particular, $g^\varepsilon(x, \rho)$ defined from $g^\varepsilon(t, x)$ similarly to $h^\varepsilon(t, \rho)$ always satisfies $g^\varepsilon(t, \rho) \in [0, 1]$ a.s. and $g^\varepsilon(t, \rho) \equiv h^\varepsilon(t, \rho)$ modulo 1.

In the next lemma, the initial distribution of $h^\varepsilon(0, \cdot)$ is taken to be $\pi \otimes \nu^\varepsilon$, where π is a uniform measure on $[0, 1]$, under the decomposition of the height:

$$(3.1) \quad g \mapsto (g(\rho), \{g(x) - g(\rho); x \in \mathbb{R}\}),$$

into the height averaged by ρ and the tilt variable. Then, $g^\varepsilon(t)$ considered as a $[0, 1] \times \tilde{\mathcal{C}}$ -valued process under the map (3.1) is stationary in t :

Lemma 3.1. *The probability measure $\pi \otimes \nu^\varepsilon$ on $[0, 1] \times \tilde{\mathcal{C}}$ is invariant under $g^\varepsilon(t, x)$.*

Proof. Take a periodic and smooth function f on $[0, 1]$ and set $\Psi(h) = f(h(\rho))$ for $h \in \mathcal{C} \cong [0, 1] \times \tilde{\mathcal{C}}$ under the map (3.1), where $h(\rho) = \int_{\mathbb{R}} h(x) \rho(x) dx$. Then, since

$$D\Psi(x; h) = f'(h(\rho))\rho(x), \quad D^2\Psi(x_1, x_2; h) = f''(h(\rho))\rho(x_1)\rho(x_2),$$

we have that

$$L_{\nabla h}^\varepsilon f(h(\rho)) := \mathcal{L}^\varepsilon \Psi(h) = \frac{\xi_\rho^\varepsilon}{2} f''(h(\rho)) + \frac{1}{2} b^\varepsilon(\nabla h) f'(h(\rho)),$$

where

$$\begin{aligned} \xi_\rho^\varepsilon &= \int_{\mathbb{R}^2} \rho(x_1) \rho(x_2) \eta_2^\varepsilon(x_1 - x_2) dx_1 dx_2, \\ b^\varepsilon(\nabla h) &= h(\rho'') + \int_{\mathbb{R}} ((\partial_x h)^2 - \xi^\varepsilon) * \eta_2^\varepsilon(x) \rho(x) dx. \end{aligned}$$

Note that $b^\varepsilon(\nabla h)$ is a tilt variable. Take another function $\Phi = \Phi(\nabla h)$ of tilt variables $\nabla h = \{\partial_x h; x \in \mathbb{R}\}$. Then, since

$$\begin{aligned} \mathcal{L}^\varepsilon(\Psi\Phi) &= \Phi \mathcal{L}^\varepsilon \Psi + \Psi \mathcal{L}^\varepsilon \Phi + \int_{\mathbb{R}^2} D\Psi(x_1; h) D\Phi(x_2; h) \eta_2^\varepsilon(x_1 - x_2) dx_1 dx_2 \\ &= \Phi L_{\nabla h}^\varepsilon f + f \mathcal{L}^\varepsilon \Phi + f'(h(\rho)) \langle D\Phi(\cdot; h) * \eta_2^\varepsilon, \rho \rangle, \end{aligned}$$

noting that $\mathcal{L}^\varepsilon \Phi$ and $\langle D\Phi(\cdot; h) * \eta_2^\varepsilon, \rho \rangle$ are tilt variables, we have that

$$(3.2) \quad \begin{aligned} \int_{[0,1] \times \tilde{\mathcal{C}}} \mathcal{L}^\varepsilon(\Psi\Phi) d\pi \otimes \nu^\varepsilon &= E^{\nu^\varepsilon} \left[\Phi(\nabla h) \int_0^1 L_{\nabla h}^\varepsilon f(h(\rho)) d\pi \right] \\ &+ \int_0^1 f(h(\rho)) d\pi E^{\nu^\varepsilon} [\mathcal{L}^\varepsilon \Phi] + \int_0^1 f'(h(\rho)) d\pi E^{\nu^\varepsilon} [\langle D\Phi(\cdot; h) * \eta_2^\varepsilon, \rho \rangle]. \end{aligned}$$

However, we easily see that

$$\int_0^1 L_{\nabla h}^\varepsilon f(h(\rho)) d\pi = \int_0^1 L_{\nabla h}^\varepsilon f(a) da = 0$$

for all fixed ∇h by the periodicity of f , and also

$$\int_0^1 f'(h(\rho)) d\pi = \int_0^1 f'(a) da = 0.$$

Moreover, noting that \mathcal{L}^ε acting on $\Phi = \Phi(\nabla h)$ through h coincides with $\mathcal{L}^{\varepsilon, U}$ acting on $\Phi = \Phi(u)$ through u , Corollary 2.10 shows that the second term in the right hand side of (3.2) vanishes, and therefore we have that

$$\int_{[0,1] \times \tilde{\mathcal{C}}} \mathcal{L}^\varepsilon(\Psi\Phi) d\pi \otimes \nu^\varepsilon = 0.$$

This can be extended to linear combinations of the functions of the form $\Psi\Phi$, and concludes the proof of the lemma. \square

We next introduce the Cole-Hopf transform $Y^\varepsilon(t, x) = e^{g^\varepsilon(t, x)}$ of the wrapped process $g^\varepsilon(t, x)$. The initial distribution of $h^\varepsilon(0, \cdot)$ is taken as mentioned above Lemma 3.1. $Y^\varepsilon(t, x)$ is called a wrapped process of $Z^\varepsilon(t, x) = e^{h^\varepsilon(t, x)}$ and satisfies $Y_\rho^\varepsilon(t) \in [1, e]$ a.s., where we define

$$(3.3) \quad Y_\rho = \exp \left\{ \int_{\mathbb{R}} \log Y(x) \rho(x) dx \right\},$$

for $Y = \{Y(x) > 0; x \in \mathbb{R}\}$ and $Y_\rho^\varepsilon(t) = (Y^\varepsilon(t))_\rho$.

Lemma 3.2. *$Y^\varepsilon(t, x)$ satisfies the following equation in generalized functions' sense:*

$$(3.4) \quad \begin{aligned} Y^\varepsilon(t, x) = Y^\varepsilon(0, x) &+ \frac{1}{2} \int_0^t \partial_x^2 Y^\varepsilon(s, x) ds + \int_0^t A^\varepsilon(x, Y^\varepsilon(s)) ds \\ &+ \int_0^t Y^\varepsilon(s, x) dW^\varepsilon(s, x) + N^\varepsilon(t, x), \end{aligned}$$

where

$$(3.5) \quad A^\varepsilon(x, Y) = \frac{1}{2} Y(x) \left\{ \left(\frac{\partial_x Y}{Y} \right)^2 * \eta_2^\varepsilon(x) - \left(\frac{\partial_x Y}{Y} \right)^2(x) \right\},$$

$$(3.6) \quad N^\varepsilon(t, x) = \int_0^t \left\{ (e - 1) 1_{\{Y_\rho^\varepsilon(s-) = 1\}} + (e^{-1} - 1) 1_{\{Y_\rho^\varepsilon(s-) = e\}} \right\} Y^\varepsilon(s-, x) N^\varepsilon(ds).$$

Proof. Note that $N^\varepsilon(t) = -[h^\varepsilon(t, \rho)]$ is expressed as

$$N^\varepsilon(t) = \int_0^t \sum_{a=\pm 1} a n^\varepsilon(ds, a)$$

with a certain point process $n^\varepsilon(ds, a)$ on $\mathbb{X} = \{\pm 1\}$. Thus, applying Itô's formula for $Y^\varepsilon(t, x) = Z^\varepsilon(t, x)e^{N^\varepsilon(t)} \equiv F(Z^\varepsilon(t, x), N^\varepsilon(t))$ with $F(z, n) = ze^n$, $z \in \mathbb{R}$, $n \in \mathbb{Z}$, we have that

$$\begin{aligned} Y^\varepsilon(t, x) &= Y^\varepsilon(0, x) + \int_0^t \frac{\partial F}{\partial z}(Z^\varepsilon(s, x), N^\varepsilon(s)) dZ^\varepsilon(s, x) \\ &\quad + \int_0^{t+} \sum_{a=\pm 1} \{F(Z^\varepsilon(s, x), N^\varepsilon(s-) + a) - F(Z^\varepsilon(s, x), N^\varepsilon(s-))\} n^\varepsilon(ds, a) \\ &= Y^\varepsilon(0, x) + \int_0^t e^{N^\varepsilon(s)} dZ^\varepsilon(s, x) + N^\varepsilon(t, x), \end{aligned}$$

where $N^\varepsilon(t, x)$ is defined by (3.6). The conclusion follows from (1.10). \square

3.3 Asymptotic behavior of the nonlinear term in (3.4)

We need to analyze the limit of the third term in the right hand side of (3.4) as $\varepsilon \downarrow 0$ at least in the stationary situation. The goal of this subsection is to show the following theorem, by which one can replace $A^\varepsilon(x, Y^\varepsilon(s))$ with a linear function $\frac{1}{24}Y^\varepsilon(s, x)$ if $h^\varepsilon(0, \cdot)$ is distributed under $\pi \otimes \nu^\varepsilon$.

Theorem 3.3. *For every $\varphi \in C_0(\mathbb{R})$ satisfying $\text{supp } \varphi \cap \text{supp } \rho = \emptyset$ (so that $\text{dist}(\text{supp } \varphi, \text{supp } \rho) > 0$), we have that*

$$\lim_{\varepsilon \downarrow 0} E^{\pi \otimes \nu^\varepsilon} \left[\sup_{0 \leq t \leq T} \left\{ \int_0^t \hat{A}^\varepsilon(\varphi, Y^\varepsilon(s)) ds \right\}^2 \right] = 0,$$

where

$$\begin{aligned} \hat{A}^\varepsilon(\varphi, Y) &= \int_{\mathbb{R}} \hat{A}^\varepsilon(x, Y) \varphi(x) dx, \\ \hat{A}^\varepsilon(x, Y) &= A^\varepsilon(x, Y) - \frac{1}{24}Y(x). \end{aligned}$$

In particular, under the time average, $A^\varepsilon(\varphi, Y^\varepsilon(s))$ can be replaced by $\frac{1}{24} \int_{\mathbb{R}} Y^\varepsilon(s, x) \varphi(x) dx$ in $L^2(\Omega)$ in a strong topology as $\varepsilon \downarrow 0$ under the equilibrium situation, if φ satisfies the above conditions.

Remark 3.1. (1) The time average is essential to show this theorem. At each fixed time, we never have this type of statement; see Remark 3.2 below.
(2) The constant $\frac{1}{24}$ frequently appears in KPZ computations; see e.g. Theorem 2.3 of [2], Theorem 1.1 of [4] and Proposition 5.1 of [3].

The proof of Theorem 3.3 will be carried out at the level of the height processes $g^\varepsilon(t, x)$ or $h^\varepsilon(t, x)$ not at that of the transformed processes $Y^\varepsilon(t, x)$ or $Z^\varepsilon(t, x)$, and in a similar way to that of the Boltzmann-Gibbs principle, which is needed in the study of the equilibrium fluctuation and establishes a replacement of a certain complex term by a linear term. In particular, we deduce an equilibrium dynamic problem into a static problem. To this end, we first consider the symmetric part $\mathcal{S}^\varepsilon := \frac{1}{2}(\mathcal{L}^\varepsilon + \mathcal{L}^{\varepsilon*})$ of the generator \mathcal{L}^ε of the height process and the corresponding Dirichlet form. Since $\mathcal{L}^\varepsilon = \mathcal{L}_0^\varepsilon + \mathcal{A}^\varepsilon$, and $\mathcal{L}_0^\varepsilon$ is symmetric

and \mathcal{A}^ε is asymmetric with respect to $\pi \otimes \nu^\varepsilon$, we see that $\mathcal{S}^\varepsilon = \mathcal{L}_0^\varepsilon$; see Corollary 2.10 (at least at the level of tilt variables) and arguments in the proof of Lemma 3.1.

The corresponding Dirichlet form is given in the next lemma.

Lemma 3.4.

$$\|\Phi\|_{1,\varepsilon}^2 := \langle \Phi, (-\mathcal{L}_0^\varepsilon)\Phi \rangle_{\pi \otimes \nu^\varepsilon} = \frac{1}{2} E^{\pi \otimes \nu^\varepsilon} \left[\int_{\mathbb{R}} (D\Phi(\cdot; h) * \eta^\varepsilon)^2(x) dx \right].$$

Before giving the proof of this lemma, we note that the limit as $\varepsilon \downarrow 0$ of ν^ε for tilt variables (and therefore defined on $\tilde{\mathcal{C}}$) can be identified with the Gaussian random measure ν on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ determined from dB . More precisely, under ν , random variables $\{X(A); A \in \mathcal{B}(\mathbb{R})\}$ are given and

- (1) $X(A) \stackrel{\text{law}}{=} N(0, |A|)$ with $|A|$ = the Lebesgue measure of A ,
- (2) If $\{A_i \in \mathcal{B}(\mathbb{R})\}_{i=1}^n$ are disjoint, then $\{X(A_i)\}_{i=1}^n$ are independent and $X(\cup_{i=1}^n A_i) = \sum_{i=1}^n X(A_i)$ a.s.

Such $X(A)$ can be constructed from $X((a, b]) := B(b) - B(a)$ in terms of the two-sided Brownian motion $\{B(x); x \in \mathbb{R}\}$ satisfying, for instance, $B(0) = 0$.

Proof of Lemma 3.4. We first note that \mathcal{L}_0 defined as the limit of $\mathcal{L}_0^\varepsilon$ as $\varepsilon \downarrow 0$, that is,

$$\mathcal{L}_0\Phi(h) = \frac{1}{2} \int_{\mathbb{R}} D^2\Phi(x, x; h) dx + \frac{1}{2} \int_{\mathbb{R}} \partial_x^2 h(x) D\Phi(x; h) dx,$$

is the generator of the Ornstein-Uhlenbeck process determined by the SPDE

$$\partial_t h = \frac{1}{2} \partial_x^2 h + \dot{W}(t, x), \quad x \in \mathbb{R},$$

and $\pi \otimes \nu$ is reversible under the wrapped process $g(t, x)$ of $h(t, x)$ so that it is reversible under \mathcal{L}_0 . It is easy to see that

$$(3.7) \quad \langle \Psi, (-\mathcal{L}_0)\Phi \rangle_{\pi \otimes \nu} = \frac{1}{2} E^{\pi \otimes \nu} \left[\int_{\mathbb{R}} D\Psi(x; h) D\Phi(x; h) dx \right].$$

Now, for a given Φ , we set $\tilde{\Phi}^\varepsilon(h) := \Phi(h * \eta^\varepsilon)$ and take $\tilde{\Phi}^\varepsilon$ and $\tilde{\Psi}^\varepsilon$ in place of Φ and Ψ , respectively, in (3.7). Then, noting that

$$\begin{aligned} D\tilde{\Phi}^\varepsilon(x; h) &= D\Phi(\cdot; h * \eta^\varepsilon) * \eta^\varepsilon(x) \\ D^2\tilde{\Phi}^\varepsilon(x_1, x_2; h) &= D^2\Phi(\cdot, \cdot; h * \eta^\varepsilon) * (\eta^\varepsilon)^{\otimes 2}(x_1, x_2), \end{aligned}$$

we can show that $\mathcal{L}_0\tilde{\Phi}^\varepsilon(h) = \mathcal{L}_0^\varepsilon\Phi(h * \eta^\varepsilon)$ and therefore $\langle \tilde{\Psi}^\varepsilon, (-\mathcal{L}_0)\tilde{\Phi}^\varepsilon \rangle_{\pi \otimes \nu} = \langle \Psi, (-\mathcal{L}_0^\varepsilon)\Phi \rangle_{\pi \otimes \nu^\varepsilon}$ by the change of variables. On the other hand, the right hand side of (3.7) with $\tilde{\Phi}^\varepsilon$ and $\tilde{\Psi}^\varepsilon$ in place of Φ and Ψ , respectively, is rewritten as

$$\begin{aligned} & \frac{1}{2} E^{\pi \otimes \nu} \left[\int_{\mathbb{R}} D\Psi(\cdot; h * \eta^\varepsilon) * \eta^\varepsilon(x) D\Phi(\cdot; h * \eta^\varepsilon) * \eta^\varepsilon(x) dx \right] \\ &= \frac{1}{2} E^{\pi \otimes \nu^\varepsilon} \left[\int_{\mathbb{R}} D\Psi(\cdot; h) * \eta^\varepsilon(x) D\Phi(\cdot; h) * \eta^\varepsilon(x) dx \right], \end{aligned}$$

by the change of variables again. This concludes the proof of the lemma. \square

The basic tool of the proof of Theorem 3.3 is the Wiener-Itô expansion. Recall that the multiple Wiener integral of order $n \geq 1$ with a kernel $\varphi_n \in \hat{L}^2(\mathbb{R}^n)$, i.e. $\varphi_n \in L^2(\mathbb{R}^n)$ and symmetric in n -variables, is defined by

$$\begin{aligned} I(\varphi_n) &= \frac{1}{n!} \int_{\mathbb{R}^n} \varphi_n(x_1, \dots, x_n) dB(x_1) \cdots dB(x_n) \\ &= \int_{-\infty}^{\infty} dB(x_1) \int_{-\infty}^{x_1} dB(x_2) \cdots \int_{-\infty}^{x_{n-1}} \varphi_n(x_1, \dots, x_n) dB(x_n), \end{aligned}$$

where B is the two-sided Brownian motion on \mathbb{R} introduced above. Set $\mathcal{H}_n = \{I(\varphi_n) \in L^2(\tilde{\mathcal{C}}, \nu); \varphi_n \in \hat{L}^2(\mathbb{R}^n)\}$ for $n \geq 1$ and $\mathcal{H}_0 = \{\text{const}\}$. Then, the well-known Wiener-Itô (Wiener chaos) expansion of $\Phi \in \mathcal{H} := L^2(\tilde{\mathcal{C}}, \nu)$ is given by

$$(3.8) \quad \Phi = \sum_{n=0}^{\infty} I(\varphi_n) \in \bigoplus_{n=0}^{\infty} \mathcal{H}_n,$$

with some $\varphi_0 \in \mathbb{R}$ and $\varphi_n \in \hat{L}^2(\mathbb{R}^n)$, where $I(\varphi_0) = \varphi_0$, and

$$(3.9) \quad \|\Phi\|_{L^2(\nu)}^2 = \sum_{n=0}^{\infty} \|I(\varphi_n)\|_{L^2(\nu)}^2 = \sum_{n=0}^{\infty} \frac{1}{n!} \|\varphi_n\|_{L^2(\mathbb{R}^n)}^2$$

holds because of the orthogonality and then by Itô isometry. The expansion (3.8) identifies $\Phi \in L^2(\tilde{\mathcal{C}}, \nu)$ with the element $\varphi = \{\varphi_n\}_{n=0}^{\infty} \in \bigoplus_{n=0}^{\infty} \hat{L}^2(\mathbb{R}^n)$ of the symmetric Fock space. The reason to do this is that it gives an explicit representation of D : $D\Phi(x)$ has representation $\{D\varphi_n\}_{n=1}^{\infty}$ where $D\varphi_n \in \hat{L}^2(\mathbb{R}^{n-1})$ is given by

$$\begin{aligned} (3.10) \quad D\varphi_n(x; x_1, \dots, x_{n-1}) &= -\frac{1}{n} \sum_{i=1}^n \partial_i \varphi_n(x_1, \dots, x_{i-1}, x, x_i, \dots, x_{n-1}) \\ &= -\partial_1 \varphi_n(x, x_1, \dots, x_{n-1}). \end{aligned}$$

The factor $\frac{1}{n}$ arises when we replace $\frac{1}{n!}$ with $\frac{1}{(n-1)!}$, and the second equality is due to the symmetry of φ_n .

The next task toward the proof of Theorem 3.3 is to express the norm $\|\Phi\|_{1,\varepsilon}$ of $\Phi \in L^2(\tilde{\mathcal{C}}, \nu)$ in terms of its Wiener chaos expansion (3.8).

Lemma 3.5. *For $\Phi \in L^2(\tilde{\mathcal{C}}, \nu)$,*

$$\|\Phi\|_{1,\varepsilon}^2 = \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{R}^{n+1}} (D\varphi_{n+1}(x; x_1, \dots, x_n) * (\eta^\varepsilon)^{\otimes(n+1)})^2 dx dx_1 \cdots dx_n.$$

Proof. Lemma 3.4 applied for a function Φ of tilt variables gives

$$\|\Phi\|_{1,\varepsilon}^2 = \frac{1}{2} \int_{\mathbb{R}} E^\nu \left[(D\Phi(\cdot; B * \eta^\varepsilon) * \eta^\varepsilon)^2(x) \right] dx.$$

Here, we see that

$$(D\Phi(\cdot; B * \eta^\varepsilon) * \eta^\varepsilon)(x)$$

$$= \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \int_{\mathbb{R}^{n-1}} D\varphi_n(x; x_1, \dots, x_{n-1}) * (\eta^\varepsilon)^{\otimes n} dB(x_1) \cdots dB(x_{n-1}).$$

Therefore, the conclusion follows from (3.9). \square

We are now almost ready to start the proof of Theorem 3.3. But, before starting, we slightly extend Lemma 2.4 of [17], p.48 stated for temporally homogeneous functions $V(x)$ to more general temporally inhomogeneous $V(s, x)$ as follows. Recall that this lemma holds generically under the stationary situation: L is the generator of a process X_t , π is its invariant probability measure, $S = (L + L^*)/2$ and semi-norms $\|\cdot\|_{-1}$ and $\|\cdot\|_1$ are defined based on the operator S : $\|f\|_1^2 = E^\pi[f \cdot (-S)f]$ and $\|f\|_{-1}^2 = \sup_g \{2E^\pi[f g] - \|g\|_1^2\}$. This extension is actually needed only for the proof of Lemma 3.12, and not for that of Theorem 3.3.

Lemma 3.6. *For $V = V(s, x)$,*

$$E^\pi \left[\sup_{0 \leq t \leq T} \left(\int_0^t V(s, X_s) ds \right)^2 \right] \leq 24 \int_0^T \|V(s, \cdot)\|_{-1}^2 ds.$$

Proof. We give the outline of the proof. For $f = f(t, x)$, let M_t be the martingale

$$(3.11) \quad M_t = f(t, X_t) - f(0, X_0) - \int_0^t \left(\frac{\partial}{\partial s} + L \right) f(s, X_s) ds.$$

Then, we have

$$(3.12) \quad E^\pi[M_t^2] = 2 \int_0^t \|f(s, \cdot)\|_1^2 ds.$$

In fact, the proof of (3.12) is similar to that of (2.16) in [17], p.47. Note that, because of the temporal inhomogeneity in our situation, two terms $E^\pi[f(t, X_t)^2] - E^\pi[f(0, X_0)^2]$ and $E^\pi[\int_0^t \frac{\partial}{\partial s} f(s, X_s)^2 ds] (= \int_0^t \frac{\partial}{\partial s} E^\pi[f(s, X_s)^2] ds)$ appear, but these terms just cancel.

For given $V(s, \cdot)$, let us take $f(s, \cdot)$ such that $E^\pi[(Sf(s, \cdot) - V(s, \cdot))^2] \leq \delta$ and $\|f(s, \cdot)\|_1 \leq \|V(s, \cdot)\|_{-1} + \delta$ for $\delta > 0$ and $s \in [0, T]$, and define M_t as in (3.11) and M_t^- by

$$M_t^- = f(T-t, X_{T-t}) - f(T, X_T) - \int_0^t \left(-\frac{\partial}{\partial s} + L^* \right) f(T-s, X_{T-s}) ds, \quad t \in [0, T].$$

Then, by a simple computation, we have

$$\begin{aligned} M_t + M_T^- - M_{T-t}^- &= - \int_0^t (L + L^*) f(s, X_s) ds \\ &= -2 \int_0^t \{V(s, X_s) - R(s, X_s)\} ds, \end{aligned}$$

where $R := V - Sf$ satisfies $E^\pi[R^2(s, \cdot)] \leq \delta$. This combined with (3.12) implies the concluding estimate as in [17]. \square

Due to this lemma, we have the bound:

$$(3.13) \quad \begin{aligned} & E^{\pi \otimes \nu^\varepsilon} \left[\sup_{0 \leq t \leq T} \left\{ \int_0^t \hat{A}^\varepsilon(\varphi(s, \cdot), Y^\varepsilon(s)) ds \right\}^2 \right] \\ & \leq 24T \sup_{\Phi \in L^2(\pi \otimes \nu^\varepsilon)} \left\{ 2E^{\pi \otimes \nu^\varepsilon} [\hat{A}^\varepsilon(\varphi, Y)\Phi] - \|\Phi\|_{1,\varepsilon}^2 \right\}. \end{aligned}$$

In fact, the reason we introduced the wrapped process mostly lies in applying this bound. For $\Phi = \Phi(h(\rho), \nabla h) \in L^2(\pi \otimes \nu^\varepsilon)$, we can rewrite

$$(3.14) \quad 2E^{\pi \otimes \nu^\varepsilon} [\hat{A}^\varepsilon(\varphi, Y)\Phi] = E^\pi [Y_\rho E^{\nu^\varepsilon} [B^\varepsilon(\varphi, Y)\Phi(h(\rho), \nabla h)]] ,$$

where

$$\begin{aligned} B^\varepsilon(x, Y) &= \left\{ \left(\frac{\partial_x Y}{Y} \right)^2 * \eta_2^\varepsilon(x) - \left(\frac{\partial_x Y}{Y} \right)^2(x) - \frac{1}{12} \right\} \frac{Y(x)}{Y_\rho} \left(= \frac{2\hat{A}^\varepsilon(x, Y)}{Y_\rho} \right), \\ B^\varepsilon(\varphi, Y) &= \int_{\mathbb{R}} B^\varepsilon(x, Y) \varphi(x) dx, \end{aligned}$$

and recall that $Y_\rho \equiv e^{h(\rho)}$, $h(x) = \log Y(x)$ is defined by (3.3). The integration of $\Phi(h(\rho), \nabla h)$ under ν^ε is performed in ∇h by fixing $h(\rho)$. Note that $B^\varepsilon(x, Y)$ is a tilt variable, though $\hat{A}^\varepsilon(x, Y)$ is not. The bound (3.13) reduces the equilibrium dynamic problem into a static problem.

The key for the proof of Theorem 3.3 is the following static bound:

Proposition 3.7. *For $\Phi = \Phi(\nabla h) \in L^2(\tilde{\mathcal{C}}, \nu)$ such that $\|\Phi\|_{1,\varepsilon} < \infty$, and $\varphi \in C_0(\mathbb{R})$ satisfying the condition of Theorem 3.3, we have that*

$$(3.15) \quad |E^{\nu^\varepsilon} [B^\varepsilon(\varphi, Y)\Phi]| \leq C(\varphi) \sqrt{\varepsilon} \|\Phi\|_{1,\varepsilon},$$

for all $0 < \varepsilon < 1 \wedge (\frac{1}{4} \text{dist}(\text{supp } \varphi, \text{supp } \rho))$ with some positive constant $C(\varphi)$, which depends only on $\|\varphi\|_\infty$ and the size of $\text{supp } \varphi$. In particular, taking $\Phi = 1$, we see $E^{\nu^\varepsilon} [B^\varepsilon(\varphi, Y)] = 0$.

As we pointed out, we will work with the height processes and not with the Cole-Hopf transformed processes. In this respect, $(\frac{\partial_x Y}{Y})^2 - \xi^\varepsilon$ is transformed back to $(\partial_x h)^2 - \xi^\varepsilon$. However, recalling that $\partial_x h = \partial_x(B * \eta^\varepsilon)$ under ν^ε in the stationary situation, by Itô's formula, we have that

$$(3.16) \quad \begin{aligned} (\partial_x h)^2 &= \left\{ \int_{\mathbb{R}} \eta^\varepsilon(x - y) dB(y) \right\}^2 \\ &= \Psi^\varepsilon(x) + \int_{\mathbb{R}} \eta^\varepsilon(x - y)^2 dy = \Psi^\varepsilon(x) + \xi^\varepsilon, \end{aligned}$$

where $\Psi^\varepsilon(x)$ is a Wiener functional of second order given by

$$(3.17) \quad \Psi^\varepsilon(x) = \int_{\mathbb{R}^2} \eta^\varepsilon(x - x_1) \eta^\varepsilon(x - x_2) dB(x_1) dB(x_2)$$

$$\equiv 2 \int_{x_1 < x_2} \eta^\varepsilon(x - x_1) \eta^\varepsilon(x - x_2) dB(x_1) dB(x_2).$$

Therefore, $(\frac{\partial_x Y}{Y})^2 - \xi^\varepsilon = \Psi^\varepsilon(x)$ and $A^\varepsilon(x, Y)$ is rewritten as

$$(3.18) \quad \frac{1}{2} Y(x) \{ \Psi^\varepsilon * \eta_2^\varepsilon(x) - \Psi^\varepsilon(x) \}.$$

Remark 3.2. (1) The term $\Psi^\varepsilon * \eta_2^\varepsilon(x) - \Psi^\varepsilon(x)$ does not converge in a strong sense. In fact, if we take $\eta(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ for simplicity, then an explicit computation shows that

$$E^{\nu^\varepsilon} [\{ \Psi^\varepsilon * \eta_2^\varepsilon(x) - \Psi^\varepsilon(x) \}^2] = \frac{1}{\pi \varepsilon^2} \left(\frac{1}{4\sqrt{5}} - \frac{1}{2\sqrt{3}} + \frac{1}{4} \right).$$

(2) We easily have that

$$E^{\nu^\varepsilon} [\Psi^\varepsilon(x)^2] = \frac{1}{\varepsilon^2} \eta_2(0)^2.$$

Comparing with (1), we see that taking the difference does not improve the magnitude in ε .

(3) The term $\Psi^\varepsilon * \eta_2^\varepsilon(x) - \Psi^\varepsilon(x)$ converges to 0 in a weak sense. More precisely, for every $\Phi \in L^2(\nu)$ whose second order Wiener chaos has a continuous kernel $\varphi_2 \in C(\mathbb{R}^2) \cap \hat{L}^2(\mathbb{R}^2)$, we have that

$$\lim_{\varepsilon \downarrow 0} E^{\nu^\varepsilon} [\{ \Psi^\varepsilon * \eta_2^\varepsilon(x) - \Psi^\varepsilon(x) \} \Phi] = 0.$$

However, this is not sufficient to analyze the limit of (3.18) because of the extra $Y(x)$.

(4) If one could have a bound on $\Psi^\varepsilon * \eta_2^\varepsilon(x) - \Psi^\varepsilon(x)$ in a Sobolev norm $\|\cdot\|_{H_r^{-\alpha}}$ with possibly $\alpha < 1/2$, then one might be able to control the limit of (3.18). However, we only have that

$$E^{\nu^\varepsilon} \left[\left\| \int_{\mathbb{R}} \{ \Psi^\varepsilon * \eta_2^\varepsilon(x) - \Psi^\varepsilon(x) \} \psi(x) dx \right\| \right] \leq C \varepsilon \|\partial_x^2 \psi\|_\infty,$$

for every $\psi \in C_b^2(\mathbb{R})$. This (with interpolation) roughly implies

$$E^{\nu^\varepsilon} \left[\|\Psi^\varepsilon * \eta_2^\varepsilon(x) - \Psi^\varepsilon(x)\|_{H_r^{-\alpha}(\mathbb{R})} \right] \leq C \varepsilon^{\alpha-1},$$

which is expected to converge to 0 only if $\alpha > 1$. Under the multiplication of $Y^\varepsilon(x)$, which is roughly in $C^{\frac{1}{2}-\delta}$ (uniformly in ε), the convergence of (3.18) to 0 cannot be expected.

(5) Proposition 3.7 shows that $\|B^\varepsilon(\varphi, Y)\|_{-1, \varepsilon} \leq C(\varphi) \sqrt{\varepsilon}$. For its L^2 -norm, we only have $\|B^\varepsilon(\varphi, Y)\|_{L^2(\Omega)} \leq C(\varphi) \varepsilon^{-1/2}$. Indeed, in the proof of the proposition stated below, we can estimate $|f_{n+2} - f_{n+2}| \leq |f_{n+2}| + |f_{n+2}|$ to avoid to have their derivatives, but we lose the factor ε in doing so.

The results mentioned in Remark 3.2 are not useful in our situation because of the extra term $Y(x)$. Since $Y(x)$ is not a tilt variable, we need to consider $\frac{Y(x)}{Y_\rho}$ instead as in $B^\varepsilon(x, Y)$.

Once Proposition 3.7 is shown, (3.14) with $\Phi \equiv 1$ implies that $E^{\pi \otimes \nu^\varepsilon} [\hat{A}^\varepsilon(\varphi, Y)] = 0$. This means that we may assume $\Phi \in L^2(\pi \otimes \nu^\varepsilon)$ in the right hand side of (3.13) satisfies $E^{\pi \otimes \nu^\varepsilon} [\Phi] = 0$. For such Φ , we have that

$$\left| 2E^{\pi \otimes \nu^\varepsilon} [\hat{A}^\varepsilon(\varphi, Y) \Phi] \right| \leq E^\pi [Y_\rho C(\varphi) \sqrt{\varepsilon} \|\Phi(h(\rho), \cdot)\|_{1, \varepsilon}]$$

$$\begin{aligned}
&\leq e C(\varphi) \sqrt{\varepsilon} \int_0^1 d\pi \left\{ \frac{1}{2} E^{\nu^\varepsilon} \left[\int_{\mathbb{R}} (D\Phi(\cdot; h(\rho), \nabla h) * \eta^\varepsilon)^2(x) dx \right] \right\}^{1/2} \\
&\leq e C(\varphi) \sqrt{\varepsilon} \left\{ \frac{1}{2} E^{\pi \otimes \nu^\varepsilon} \left[\int_{\mathbb{R}} (D\Phi(\cdot; h(\rho), \nabla h) * \eta^\varepsilon)^2(x) dx \right] \right\}^{1/2} \\
&\leq e C(\varphi) \sqrt{\varepsilon} \|\Phi\|_{1,\varepsilon},
\end{aligned}$$

where the operator D acts only on the tilt variable ∇h . We have used Proposition 3.7, Lemma 3.4 and $Y_\rho \in [1, e]$ for the second line, Schwarz's inequality for the third line and Lemma 3.8 stated below for the fourth line. Therefore, the right hand side of (3.13) is bounded by

$$24T \sup_{\alpha \in \mathbb{R}} \{eC(\varphi) \sqrt{\varepsilon} \alpha - \alpha^2\} = 24T \left(\frac{e}{2} C(\varphi) \sqrt{\varepsilon}\right)^2,$$

in which we write $\alpha = \|\Phi\|_{1,\varepsilon}$. This tends to 0 as $\varepsilon \downarrow 0$, and concludes the proof of Theorem 3.3.

Lemma 3.8. *For $\Phi = \Phi(h(\rho), \nabla h)$ such that $E^{\pi \otimes \nu^\varepsilon}[\Phi] = 0$,*

$$\begin{aligned}
\|\Phi\|_{1,\varepsilon}^2 &= \frac{\xi_\rho^\varepsilon}{2} E^{\pi \otimes \nu^\varepsilon} \left[\left(\frac{\partial \Phi}{\partial h(\rho)}(h(\rho), \nabla h) \right)^2 \right] \\
&\quad + \frac{1}{2} E^{\pi \otimes \nu^\varepsilon} \left[\int_{\mathbb{R}} (D\Phi(\cdot; h(\rho), \nabla h) * \eta^\varepsilon)^2(x) dx \right].
\end{aligned}$$

In the above formula, D acts only on the tilt variables and ξ_ρ^ε is defined in the proof of Lemma 3.1.

Proof. Denote D acting on $h = (h(\rho), \nabla h)$ by \tilde{D} for distinction. Then, it can be expressed as

$$\tilde{D}\Phi(x; h(\rho), \nabla h) = \frac{\partial \Phi}{\partial h(\rho)}(h(\rho), \nabla h) \rho(x) + D\Phi(x; h(\rho), \nabla h),$$

so that Lemma 3.4 implies that

$$\|\Phi\|_{1,\varepsilon}^2 = \frac{1}{2} E^{\pi \otimes \nu^\varepsilon} \left[\int_{\mathbb{R}} \left\{ \frac{\partial \Phi}{\partial h(\rho)}(h(\rho), \nabla h) \rho * \eta^\varepsilon(x) + D\Phi(\cdot; h(\rho), \nabla h) * \eta^\varepsilon(x) \right\}^2 dx \right].$$

We expand the square inside the integration, then the cross term becomes

$$(3.19) \quad E^{\pi \otimes \nu^\varepsilon} \left[\frac{\partial \Phi}{\partial h(\rho)}(h(\rho), \nabla h) \int_{\mathbb{R}} \rho * \eta^\varepsilon(x) (D\Phi(\cdot; h(\rho), \nabla h) * \eta^\varepsilon)(x) dx \right] = 0$$

and this shows the conclusion. (3.19) is shown, first for Φ of the form $\Phi = \sum_{i=1}^\ell f_i(h(\rho)) \Phi_i(\nabla h)$, where we choose $\{f_i(a) = \sqrt{2} \sin \pi i a\}_{i=1}^\infty$ which is a complete orthonormal system of $L^2([0, 1], \pi)$. Since $E^{\pi \otimes \nu^\varepsilon}[\Phi] = 0$, we may assume $E^\pi[f_i] = 0$ so that i are even. Indeed for such Φ , the left hand side of (3.19) can be rewritten as

$$\sum_{i,j=2: \text{ even}}^\ell \int_0^1 f'_i f_j d\pi E^{\nu^\varepsilon} \left[\Phi_i \int_{\mathbb{R}} \rho * \eta^\varepsilon(x) (D\Phi_j(\cdot; \nabla h) * \eta^\varepsilon)(x) dx \right].$$

However, we easily see that $\int_0^1 f'_i f_j d\pi = 0$ for all even $i, j \geq 2$ and this proves (3.19). The general $\Phi \in L^2(\pi \otimes \nu^\varepsilon)$ can be approximated by the functions of the above form. \square

Only the proof of Proposition 3.7 is left. Before giving it, we recall the diagram formula in order to compute the second order chaos in the products of two Wiener functionals $\frac{Y(x)}{Y_\rho}$ and Φ . Let $n_1, \dots, n_m \in \mathbb{N}$ be given. We call Γ the set of all diagrams γ , which are collections of (undirected) edges connecting vertices in $V := \{(j, \ell); \ell = 1, \dots, m, j = 1, \dots, n_\ell\}$, in such a way that each edge in γ connects two vertices (j_1, ℓ_1) and (j_2, ℓ_2) only when $\ell_1 \neq \ell_2$ and each vertex is attached to at most one edge. We denote $N = \sum_{\ell=1}^m n_\ell$ and the number of edges in γ by $|\gamma|$. We define the function $\varphi_\gamma \in \hat{L}^2(\mathbb{R}^{N-2|\gamma|})$ as follows: We first introduce a function φ of N variables $\{x_{j,\ell}\}$ by

$$\varphi(x_{j,\ell}, \ell = 1, \dots, m, j = 1, \dots, n_\ell) := \prod_{\ell=1}^m \varphi_\ell(x_{j,\ell}, j = 1, \dots, n_\ell).$$

We call the variables $\{x_{j,\ell}, \ell = 1, \dots, m, j = 1, \dots, n_\ell\}$ simply as $\{x_1, \dots, x_N\}$ and call φ again its symmetrization. Then, φ_γ is defined from φ by truncating the last $2|\gamma|$ -variables:

$$\varphi(x_1, \dots, x_{N-2|\gamma|}) := \int_{\mathbb{R}^{2|\gamma|}} \varphi(x_1, \dots, x_{N-2|\gamma|}, p_1, p_1, \dots, p_{|\gamma|}, p_{|\gamma|}) dp_1 \cdots dp_{|\gamma|}.$$

Then, we have the following diagram formula; see Major [19], Section 5, under a slightly different setting.

Lemma 3.9. *For $\varphi_1 \in \hat{L}^2(\mathbb{R}^{n_1}), \dots, \varphi_m \in \hat{L}^2(\mathbb{R}^{n_m})$ with $n_1, \dots, n_m \geq 1$, we have*

$$I(\varphi_1) \cdots I(\varphi_m) = \sum_{\gamma \in \Gamma} \frac{(N - 2|\gamma|)!}{n_1! \cdots n_m!} I(\varphi_\gamma).$$

We are now ready to give the proof of Proposition 3.7.

Proof of Proposition 3.7. We first notice that, under ν , $\frac{Y(x)}{Y_\rho}$ has an expression:

$$\frac{Y(x)}{Y_\rho} = e^{B(x) - \int_{\mathbb{R}} B(y) \rho(y) dy},$$

and the exponent can be rewritten as

$$B(x) - \int_{\mathbb{R}} B(y) \rho(y) dy = \int_{\mathbb{R}} \phi_x(u) dB(u),$$

where

$$(3.20) \quad \phi_x(u) = 1_{(-\infty, x]}(u) + \theta(u), \quad \theta(u) = - \int_u^\infty \rho(y) dy.$$

Note that, from the condition of ρ , $\phi_x(\cdot)$ has a compact support: $\text{supp } \phi_x \subset [x \wedge (-1), x \vee 1]$, $|\phi_x(u)| \leq 1$, has a jump only at $u = x$ and θ is smooth. Therefore, $\frac{Y(x)}{Y_\rho}$ has the following Wiener-Itô expansion under ν :

$$(3.21) \quad \frac{Y(x)}{Y_\rho} = e^{a(x)} \left\{ 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\mathbb{R}^n} \phi_x^{\otimes n}(u_1, \dots, u_n) dB(u_1) \cdots dB(u_n) \right\},$$

where $a(x) = \frac{1}{2} \int_{\mathbb{R}} \phi_x^2(u) du$. In fact, one can apply the well-known result for the expansion of exponential martingales written, e.g., in [15], p.167 for $M_t = \int_{-\infty}^t \phi_x(u) dB(u)$, and letting $t \rightarrow \infty$.

Since Ψ^ε and $\Psi^\varepsilon * \eta^\varepsilon$ are both second order Wiener chaoses, to compute the expectation

$$(3.22) \quad E^{\nu^\varepsilon} \left[\{ \Psi^\varepsilon * \eta_2^\varepsilon(x) - \Psi^\varepsilon(x) \} \frac{Y(x)}{Y_\rho} \Phi \right],$$

what we need to obtain is the kernel $\bar{\varphi}_2(x_1, x_2)$ of the second order Wiener chaos in the product $\Phi \cdot \frac{Y(x)}{Y_\rho}$. Denoting the kernel of the n th order Wiener chaos in $\frac{Y(x)}{Y_\rho}$ except the factor $e^{a(x)}$ by $\psi_n(u_1, \dots, u_n) = \phi_x^{\otimes n}(u_1, \dots, u_n)$, in the expansion of the product

$$(3.23) \quad \Phi \cdot \frac{Y(x)}{Y_\rho} = e^{a(x)} \left(\sum_{m_1=0}^{\infty} I(\varphi_{m_1}) \right) \left(\sum_{m_2=0}^{\infty} I(\psi_{m_2}) \right) = e^{a(x)} \sum_{m_1, m_2=0}^{\infty} I(\varphi_{m_1}) I(\psi_{m_2}),$$

we apply the diagram formula to get the explicit formula for $\bar{\varphi}_2(x_1, x_2)$:

$$\begin{aligned} \bar{\varphi}_2(x_1, x_2) &= e^{a(x)} \sum_{n=0}^{\infty} \binom{n+2}{2} n! \times \frac{2!}{(n+2)!n!} \int_{\mathbb{R}^n} \varphi_{n+2}(\underline{u}, x_1, x_2) \psi_n(\underline{u}) d\underline{u} \\ &\quad + e^{a(x)} \sum_{n=0}^{\infty} \binom{n+2}{2} n! \times \frac{2!}{(n+2)!n!} \int_{\mathbb{R}^n} \varphi_n(\underline{u}) \psi_{n+2}(\underline{u}, x_1, x_2) d\underline{u} \\ &\quad + e^{a(x)} \sum_{n=0}^{\infty} (n+1)^2 n! \times \frac{2!}{(n+1)!(n+1)!} \int_{\mathbb{R}^n} \varphi_{n+1}(\underline{u}, x_1) \psi_{n+1}(\underline{u}, x_2) d\underline{u} \\ &\quad + e^{a(x)} \sum_{n=0}^{\infty} (n+1)^2 n! \times \frac{2!}{(n+1)!(n+1)!} \int_{\mathbb{R}^n} \varphi_{n+1}(\underline{u}, x_2) \psi_{n+1}(\underline{u}, x_1) d\underline{u} \\ &=: \bar{\varphi}_2^{(1)}(x_1, x_2) + \bar{\varphi}_2^{(2)}(x_1, x_2) + \bar{\varphi}_2^{(3)}(x_1, x_2) + \bar{\varphi}_2^{(4)}(x_1, x_2), \end{aligned}$$

where we denote $\underline{u} = (u_1, \dots, u_n)$ and $d\underline{u} = du_1 \cdots du_n$. Note that, to obtain the second order term, (m_1, m_2) which we need to take care are only of the forms $\{(n+2, n), (n, n+2), (n+1, n+1), n \geq 0\}$, in which case $N = 2n+2$, and we may only consider γ satisfying $N - 2|\gamma| = 2$, i.e., $|\gamma| = n$. For example, when $(m_1, m_2) = (n+2, n)$, the prefactor in the diagram formula becomes as above, since $|\Gamma| = \binom{n+2}{2} n!$, and when $(m_1, m_2) = (n+1, n+1)$, $|\Gamma| = (n+1)^2 n!$.

In the rest, we will show that the contributions of $\bar{\varphi}_2^{(1)}$, $\bar{\varphi}_2^{(3)}$, $\bar{\varphi}_2^{(4)}$ to the expectation (3.22) are small, while that of $\bar{\varphi}_2^{(2)}$ exactly cancels with $\frac{1}{12} E^{\nu^\varepsilon} \left[\frac{Y(x)}{Y_\rho} \Phi \right]$, when integrated with a test function $\varphi = \varphi(x)$.

Under ν^ε , the kernels φ_n and ψ_n are replaced by $\varphi_n * (\eta^\varepsilon)^{\otimes n}$ and $\psi_n * (\eta^\varepsilon)^{\otimes n}$, respectively, where $\varphi_n * (\eta^\varepsilon)^{\otimes n}$ is defined by

$$\varphi_n * (\eta^\varepsilon)^{\otimes n}(\underline{u}) := \int_{\mathbb{R}^n} \varphi_n(\underline{v}) \eta^\varepsilon(u_1 - v_1) \cdots \eta^\varepsilon(u_n - v_n) d\underline{v},$$

where $\underline{v} = (v_1, \dots, v_n)$ and $d\underline{v} = dv_1 \cdots dv_n$. Recall that $\Psi^\varepsilon * \eta_2^\varepsilon(x) - \Psi^\varepsilon(x)$ is a second order Wiener chaos with the kernel:

$$(3.24) \quad 2 \left\{ \int_{\mathbb{R}} \eta^\varepsilon(y - x_1) \eta^\varepsilon(y - x_2) \eta_2^\varepsilon(x - y) dy - \eta^\varepsilon(x - x_1) \eta^\varepsilon(x - x_2) \right\}.$$

Then, the contribution of the first term $\bar{\varphi}_2^{(1)}$ to the expectation (3.22) is given by, neglecting the factor $e^{a(x)} \sum_{n=0}^{\infty} \frac{1}{n!}$ (2 in (3.24) cancels with $\frac{1}{2}$ appearing in (3.9) for $n = 2$),

$$\begin{aligned} & \int_{\mathbb{R}^2} dx_1 dx_2 \left\{ \int_{\mathbb{R}^n} \left(\varphi_{n+2} * (\eta^\varepsilon)^{\otimes(n+2)} \right) (\underline{u}, x_1, x_2) \left(\phi_x^{\otimes n} * (\eta^\varepsilon)^{\otimes n} \right) (\underline{u}) d\underline{u} \right\} \\ & \quad \times \left\{ \int_{\mathbb{R}} \eta^\varepsilon(y - x_1) \eta^\varepsilon(y - x_2) \eta_2^\varepsilon(x - y) dy - \eta^\varepsilon(x - x_1) \eta^\varepsilon(x - x_2) \right\} \\ & = \int_{\mathbb{R}^n} \phi_x^{\otimes n}(\underline{u}) d\underline{u} \left\{ \int_{\mathbb{R}} \left(\varphi_{n+2} * (\eta_2^\varepsilon)^{\otimes(n+2)} \right) (\underline{u}, y, y) \eta_2^\varepsilon(x - y) dy \right. \\ & \quad \left. - \left(\varphi_{n+2} * (\eta_2^\varepsilon)^{\otimes(n+2)} \right) (\underline{u}, x, x) \right\}. \end{aligned}$$

In the above computation, we first move η^ε in $\phi_x^{\otimes n} * (\eta^\varepsilon)^{\otimes n}$ to $\varphi_{n+2} * (\eta^\varepsilon)^{\otimes(n+2)}$, which gives $(\varphi_{n+2} * (\eta_2^\varepsilon)^{\otimes n} * (\eta^\varepsilon)^{\otimes 2})(\underline{u}, x_1, x_2)$. Then, we integrate in x_1 and x_2 and obtain the above formula.

Therefore, the contribution of $\bar{\varphi}_2^{(1)}$ to the left hand side of (3.15) is given by $\sum_{n=0}^{\infty} \frac{1}{n!} I_n^{(1)}$, where

$$\begin{aligned} I_n^{(1)} &:= \int_{\mathbb{R}} \varphi(x) e^{a(x)} dx \int_{\mathbb{R}^n} \phi_x^{\otimes n}(\underline{u}) d\underline{u} \\ & \quad \times \left\{ \int_{\mathbb{R}} (f_{n+2} * (\eta^\varepsilon)^{\otimes(n+2)})(\underline{u}, y, y) \eta_2^\varepsilon(x - y) dy - (f_{n+2} * (\eta^\varepsilon)^{\otimes(n+2)})(\underline{u}, x, x) \right\}, \end{aligned}$$

with $f_{n+2} := \varphi_{n+2} * (\eta^\varepsilon)^{\otimes(n+2)}$. We have rewritten as $\varphi_{n+2} * (\eta_2^\varepsilon)^{\otimes(n+2)} = f_{n+2} * (\eta^\varepsilon)^{\otimes(n+2)}$ in the above formula. The difference in the braces in the right hand side of $I_n^{(1)}$ can be expressed as the expectation:

$$E[f_{n+2}(\underline{u} + U^\varepsilon, x + Y^\varepsilon + X_1^\varepsilon, x + Y^\varepsilon + X_2^\varepsilon) - f_{n+2}(\underline{u} + U^\varepsilon, x + X_1^\varepsilon, x + X_2^\varepsilon)],$$

where $U^\varepsilon = (U_i^\varepsilon)_{i=1}^n$ is an \mathbb{R}^n -valued random variable with independent components U_i^ε distributed under $\eta^\varepsilon(y) dy$ (η^ε in short), $X_1^\varepsilon, X_2^\varepsilon$ are \mathbb{R} -valued random variables distributed under η^ε , Y^ε is an \mathbb{R} -valued random variable distributed under η_2^ε , and all random variables are mutually independent. We estimate

$$\begin{aligned} & |f_{n+2}(\underline{u} + U^\varepsilon, x + Y^\varepsilon + X_1^\varepsilon, x + Y^\varepsilon + X_2^\varepsilon) - f_{n+2}(\underline{u} + U^\varepsilon, x + X_1^\varepsilon, x + X_2^\varepsilon)| \\ &= \left| \int_0^1 \frac{\partial}{\partial \lambda} f_{n+2}(\underline{u} + U^\varepsilon, x + \lambda Y^\varepsilon + X_1^\varepsilon, x + \lambda Y^\varepsilon + X_2^\varepsilon) d\lambda \right| \\ &= \left| \int_0^1 Y^\varepsilon \sum_{k=n+1}^{n+2} \partial_k f_{n+2}(\underline{u} + U^\varepsilon, x + \lambda Y^\varepsilon + X_1^\varepsilon, x + \lambda Y^\varepsilon + X_2^\varepsilon) d\lambda \right| \end{aligned}$$

$$\leq 4\varepsilon \int_0^1 |\partial_{n+2} f_{n+2}(\underline{u} + U^\varepsilon, x + \lambda Y^\varepsilon + X_1^\varepsilon, x + \lambda Y^\varepsilon + X_2^\varepsilon)| d\lambda, \quad \text{a.s.}$$

Note that $\text{supp } \eta \subset [-1, 1]$ implies $\text{supp } \eta_2^\varepsilon \subset [-2\varepsilon, 2\varepsilon]$ and therefore $|Y^\varepsilon| \leq 2\varepsilon$ a.s. We also used the symmetry of f_{n+2} in $(n+2)$ -variables. Accordingly, since $\varphi \in C_0(\mathbb{R})$ implies $\text{supp } \varphi \subset [-K, K]$ with some $K \geq 1$ and $\text{supp } \phi_x \subset [x \wedge (-1), x \vee 1] \subset [-K, K]$ for $x \in [-K, K]$, we have

$$\begin{aligned} |I_n^{(1)}| &\leq 4\varepsilon \|\varphi e^{a(x)}\|_\infty \int_{-K}^K dx \int_{\mathbb{R}^n} 1_{[-K, K]^n}(\underline{u}) d\underline{u} \\ &\quad \times \int_0^1 E[|\partial_{n+2} f_{n+2}(\underline{u} + U^\varepsilon, x + \lambda Y^\varepsilon + X_1^\varepsilon, x + \lambda Y^\varepsilon + X_2^\varepsilon)|] d\lambda \\ &\leq 4\varepsilon \|\varphi e^{a(x)}\|_\infty \int_{-K-3\varepsilon}^{K+3\varepsilon} dx \int_{\mathbb{R}^n} 1_{[-K-\varepsilon, K+\varepsilon]^n}(\underline{u}) d\underline{u} \\ &\quad \times E[|\partial_{n+2} f_{n+2}(\underline{u}, x, x + X_2^\varepsilon - X_1^\varepsilon)|]. \end{aligned}$$

The last line is obtained by putting the integrals in (x, \underline{u}) inside the expectation and then applying the change of variables $\underline{u}' = \underline{u} + U^\varepsilon$ and $x' = x + \lambda Y^\varepsilon + X_1^\varepsilon$. We enlarge the domains of the integrations a little bit. Since $X_2^\varepsilon - X_1^\varepsilon$ is distributed under η_2^ε , this is equal to

$$4\varepsilon \|\varphi e^{a(x)}\|_\infty \int_{\mathbb{R}^{n+2}} 1_{[-K-3\varepsilon, K+3\varepsilon]}(x) 1_{[-K-\varepsilon, K+\varepsilon]^n}(\underline{u}) |\partial_{n+2} f_{n+2}(\underline{u}, x, x + y)| d\underline{u} dx \eta_2^\varepsilon(y) dy.$$

Apply Schwarz's inequality, and we obtain that

$$\begin{aligned} |I_n^{(1)}| &\leq 4\varepsilon \|\varphi e^{a(x)}\|_\infty \left\{ \int_{\mathbb{R}^{n+2}} 1_{[-K-3\varepsilon, K+3\varepsilon]}(x) 1_{[-K-\varepsilon, K+\varepsilon]^n}(\underline{u}) d\underline{u} dx \eta_2^\varepsilon(y) dy \right\}^{1/2} \\ &\quad \times \left\{ \int_{\mathbb{R}^{n+2}} |\partial_{n+2} f_{n+2}(\underline{u}, x, x + y)|^2 d\underline{u} dx \eta_2^\varepsilon(y) dy \right\}^{1/2} \\ &\leq 4\sqrt{\varepsilon} \|\varphi e^{a(x)}\|_\infty \|\eta_2\|_\infty^{1/2} (2K+6)^{(n+1)/2} \left\{ \int_{\mathbb{R}^{n+2}} |\partial_{n+2} f_{n+2}(\underline{u}, x, y)|^2 d\underline{u} dx dy \right\}^{1/2}, \end{aligned}$$

since $0 < \varepsilon < 1$. We have used a rough estimate: $|\eta_2^\varepsilon(y)| \leq \|\eta_2\|_\infty / \varepsilon$. Thus, we obtain that

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{1}{n!} |I_n^{(1)}| &\leq C\sqrt{\varepsilon} \sum_{n=0}^{\infty} \frac{(2K+6)^{(n+1)/2}}{n!} \left\{ \int_{\mathbb{R}^{n+2}} |\partial_{n+2} f_{n+2}|^2 d\underline{x} \right\}^{1/2} \\ &\leq C\sqrt{\varepsilon} \left\{ \sum_{n=0}^{\infty} \frac{(n+1)^2 (2K+6)^{(n+1)}}{(n+1)!} \right\}^{1/2} \left\{ \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \int_{\mathbb{R}^{n+2}} |\partial_{n+2} f_{n+2}|^2 d\underline{x} \right\}^{1/2} \\ &\leq C'\sqrt{\varepsilon} \|\Phi\|_{1,\varepsilon}, \end{aligned}$$

by Schwarz's inequality and Lemma 3.5.

Next, the contribution of the second term $\bar{\varphi}_2^{(2)}$ to the expectation (3.22) is given by, again neglecting the factor $e^{a(x)} \sum_{n=0}^{\infty} \frac{1}{n!}$,

$$\int_{\mathbb{R}^2} dx_1 dx_2 \left\{ \int_{\mathbb{R}^n} (\varphi_n * (\eta^\varepsilon)^{\otimes n})(\underline{u}) \left(\phi_x^{\otimes(n+2)} * (\eta^\varepsilon)^{\otimes(n+2)} \right)(\underline{u}, x_1, x_2) d\underline{u} \right\}$$

$$\begin{aligned}
& \times \left\{ \int_{\mathbb{R}} \eta^\varepsilon(y - x_1) \eta^\varepsilon(y - x_2) \eta_2^\varepsilon(x - y) dy - \eta^\varepsilon(x - x_1) \eta^\varepsilon(x - x_2) \right\} \\
& = \int_{\mathbb{R}^n} (f_n * (\eta^\varepsilon)^{\otimes n})(\underline{u}) \phi_x^{\otimes n}(\underline{u}) d\underline{u} \\
& \times \left\{ \int_{\mathbb{R}} (\phi_x^{\otimes 2} * (\eta_2^\varepsilon)^{\otimes 2})(y, y) \eta_2^\varepsilon(x - y) dy - (\phi_x^{\otimes 2} * (\eta_2^\varepsilon)^{\otimes 2})(x, x) \right\}.
\end{aligned}$$

Here, the last term in the braces can be represented by means of expectations:

$$\begin{aligned}
& \int_{\mathbb{R}} (\phi_x^{\otimes 2} * (\eta_2^\varepsilon)^{\otimes 2})(y, y) \eta_2^\varepsilon(x - y) dy - (\phi_x^{\otimes 2} * (\eta_2^\varepsilon)^{\otimes 2})(x, x) \\
& = E[\phi_x^{\otimes 2}(x + R_1 + R_3, x + R_2 + R_3)] - E[\phi_x^{\otimes 2}(x + R_1, x + R_2)],
\end{aligned}$$

where $\{R_1, R_2, R_3\}$ are independent random variables distributed under η_2^ε . By expanding

$$\phi_x^{\otimes 2} = 1_{(-\infty, x]}^{\otimes 2} + 1_{(-\infty, x]} \otimes \theta + \theta \otimes 1_{(-\infty, x]} + \theta^{\otimes 2},$$

the above difference of two expectations can be rewritten as

$$J_1 + 2J_2^\varepsilon(x) + J_3^\varepsilon(x),$$

where

$$\begin{aligned}
J_1 &= E[1_{(-\infty, 0]}^{\otimes 2}(R_1 + R_3, R_2 + R_3)] - E[1_{(-\infty, 0]}^{\otimes 2}(R_1, R_2)] \\
J_2^\varepsilon(x) &= E[1_{(-\infty, 0]}(R_1 + R_3) \theta(x + R_2 + R_3)] - E[1_{(-\infty, 0]}(R_1) \theta(x + R_2)] \\
J_3^\varepsilon(x) &= E[\theta(x + R_1 + R_3) \theta(x + R_2 + R_3)] - E[\theta(x + R_1) \theta(x + R_2)].
\end{aligned}$$

However, we see that $\text{supp } J_2^\varepsilon \subset (\text{supp } \rho)^{4\varepsilon} := \{y \in \mathbb{R}; |y - x| \leq 4\varepsilon \text{ for some } x \in \text{supp } \rho\}$ and $\text{supp } J_3^\varepsilon \subset (\text{supp } \rho)^{4\varepsilon}$; recall that $\text{supp } \rho$ is connected and $E[1_{(-\infty, 0]}(R_1 + R_3)] = E[1_{(-\infty, 0]}(R_1)] = \frac{1}{2}$ by the symmetry of R_i for $J_2^\varepsilon(x)$. Therefore, from our assumption: $\text{dist}(\text{supp } \varphi, \text{supp } \rho) > 4\varepsilon$, $\varphi(x) J_2^\varepsilon(x) = \varphi(x) J_3^\varepsilon(x) = 0$ for all $x \in \mathbb{R}$. On the other hand, we have

$$J_1 = \frac{1}{12}.$$

In fact, by the symmetry of η_2^ε ,

$$\begin{aligned}
(3.25) \quad P(R_1 + R_3 > 0, R_2 + R_3 > 0) &= P(R_1 - R_3 > 0, R_2 - R_3 > 0) \\
&= P\left(R_3 = \min_{i=1,2,3} R_i\right) = \frac{1}{3},
\end{aligned}$$

for the first expectation and the second one is $\frac{1}{4}$ as easily seen. The constant $\frac{1}{12}$ is universal in the sense that it does not depend on the specific distributions of independent random variables $\{R_1, R_2, R_3\}$ if they are symmetric (and have densities). Summarizing these, we see that the contribution of $\bar{\varphi}_2^{(2)}$ to the expectation (3.22), when multiplied by φ , is given by

$$(3.26) \quad \frac{1}{12} e^{a(x)} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{R}^n} (f_n * (\eta^\varepsilon)^{\otimes n})(\underline{u}) \phi_x^{\otimes n}(\underline{u}) d\underline{u}.$$

However, the series in (3.26) just cancels with the expectation $E^{\nu^\varepsilon} \left[\frac{Y(x)}{Y_\rho} \Phi \right]$ divided by 12. Indeed, this expectation can be computed by picking up the 0th order term in the product (3.23), and it is again an application of the diagram formula. Indeed, we may take care (m_1, m_2) of the forms $(n, n), n \geq 0$ only, in which case $N = 2n$, and may consider only γ such that $|\gamma| = n$, i.e., diagrams connecting all vertices of the forms (j_1, ℓ_1) and (j_2, ℓ_2) with $\ell_1 \neq \ell_2$. The number of such γ 's is given by $|\Gamma| = n!$. Thus, we have

$$\begin{aligned} E^{\nu^\varepsilon} \left[\frac{Y(x)}{Y_\rho} \Phi \right] &= e^{a(x)} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{R}^n} (\varphi_n * (\eta^\varepsilon)^{\otimes n})(\underline{u}) (\phi_x^{\otimes n} * (\eta^\varepsilon)^{\otimes n})(\underline{u}) d\underline{u} \\ &= e^{a(x)} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{R}^n} (f_n * (\eta^\varepsilon)^{\otimes n})(\underline{u}) \phi_x^{\otimes n}(\underline{u}) d\underline{u}, \end{aligned}$$

which is exactly the same series in (3.26) except the factor $\frac{1}{12}$.

The contribution from the third term $\bar{\varphi}_2^{(3)}$ to the expectation (3.22) is given by, neglecting the factor $a(x) \sum_{n=0}^{\infty} \frac{2}{n!}$,

$$\begin{aligned} &\int_{\mathbb{R}^2} dx_1 dx_2 \left\{ \int_{\mathbb{R}^n} (\varphi_{n+1} * (\eta^\varepsilon)^{\otimes(n+1)})(\underline{u}, x_1) (\phi_x^{\otimes(n+1)} * (\eta^\varepsilon)^{\otimes(n+1)})(\underline{u}, x_2) d\underline{u} \right\} \\ &\quad \times \left\{ \int_{\mathbb{R}} \eta^\varepsilon(y - x_1) \eta^\varepsilon(y - x_2) \eta_2^\varepsilon(x - y) dy - \eta^\varepsilon(x - x_1) \eta^\varepsilon(x - x_2) \right\} \\ &= \int_{\mathbb{R}^n} \phi_x^{\otimes n}(\underline{u}) d\underline{u} \left\{ \int_{\mathbb{R}} (\varphi_{n+1} * (\eta_2^\varepsilon)^{\otimes(n+1)})(\underline{u}, y) (\phi_x * \eta_2^\varepsilon)(y) \eta_2^\varepsilon(x - y) dy \right. \\ &\quad \left. - (\varphi_{n+1} * (\eta_2^\varepsilon)^{\otimes(n+1)})(\underline{u}, x) (\phi_x * \eta_2^\varepsilon)(x) \right\}. \end{aligned}$$

However, we see that

$$\begin{aligned} \int_{\mathbb{R}} (\phi_x * \eta_2^\varepsilon)(y) \eta_2^\varepsilon(x - y) dy &= E[\phi_x(x + R_1 + R_2)] = \frac{1}{2} + \theta * \eta_4^\varepsilon(x), \\ \phi_x * \eta_2^\varepsilon(x) &= E[\phi_x(x + R_1)] = \frac{1}{2} + \theta * \eta_2^\varepsilon(x), \end{aligned}$$

where $\{R_1, R_2\}$ are independent random variables distributed under η_2^ε , from the symmetry of R_i .

Therefore, the contribution of $\bar{\varphi}_2^{(3)}$ in the left hand side of (3.15) is given by $\sum_{n=0}^{\infty} \frac{2}{n!} I_n^{(3)}$, where

$$\begin{aligned} I_n^{(3)} &:= \int_{\mathbb{R}} \varphi(x) e^{a(x)} dx \int_{\mathbb{R}^n} \phi_x^{\otimes n}(\underline{u}) d\underline{u} \\ &\quad \times \int_{\mathbb{R}} \left\{ f_{n+1} * (\eta^\varepsilon)^{\otimes(n+1)}(\underline{u}, y) - f_{n+1} * (\eta^\varepsilon)^{\otimes(n+1)}(\underline{u}, x) \right\} (\phi_x * \eta_2^\varepsilon)(y) \eta_2^\varepsilon(x - y) dy \\ &\quad + \int_{\mathbb{R}} \varphi(x) e^{a(x)} \{ \theta * \eta_4^\varepsilon(x) - \theta * \eta_2^\varepsilon(x) \} dx \int_{\mathbb{R}^n} f_{n+1} * (\eta^\varepsilon)^{\otimes(n+1)}(\underline{u}, x) \phi_x^{\otimes n}(\underline{u}) d\underline{u}. \end{aligned}$$

The second term in $I_n^{(3)}$ vanishes from our assumption by noting that $\text{supp}\{\theta * \eta_4^\varepsilon(x) - \theta * \eta_2^\varepsilon(x)\} \subset (\text{supp } \rho)^{4\varepsilon}$. For the first term, estimating $|(\phi_x * \eta_2^\varepsilon)(y)| \leq 1$, the absolute value

of the last integral in y can be bounded by

$$E[|f_{n+1}(\underline{u} + U^\varepsilon, x + Y^\varepsilon + X_1^\varepsilon) - f_{n+1}(\underline{u} + U^\varepsilon, x + X_1^\varepsilon)|],$$

where $U^\varepsilon, X_1^\varepsilon, Y^\varepsilon$ are the same as before, and this can be estimated further by

$$2\varepsilon \int_0^1 E[|\partial_{n+1} f_{n+1}(\underline{u} + U^\varepsilon, x + \lambda Y^\varepsilon + X_1^\varepsilon)|] d\lambda.$$

Thus,

$$\begin{aligned} |I_n^{(3)}| &\leq 2\varepsilon \|\varphi e^{a(x)}\|_\infty \int_{\mathbb{R}^{n+1}} 1_{[-K-3\varepsilon, K+3\varepsilon]}(x) 1_{[-K-\varepsilon, K+\varepsilon]}^n(\underline{u}) |\partial_{n+1} f_{n+1}(\underline{u}, x)| d\underline{u} dx \\ &\leq 2\varepsilon \|\varphi e^{a(x)}\|_\infty (2K+6)^{(n+1)/2} \left\{ \int_{\mathbb{R}^{n+1}} |\partial_{n+1} f_{n+1}(\underline{x})|^2 d\underline{x} \right\}^{1/2}, \end{aligned}$$

by Schwarz's inequality. Therefore, we get that

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{2}{n!} |I_n^{(3)}| &\leq C\varepsilon \sum_{n=0}^{\infty} \frac{1}{n!} (2K+6)^{(n+1)/2} \left\{ \int_{\mathbb{R}^{n+1}} |\partial_{n+1} f_{n+1}(\underline{x})|^2 d\underline{x} \right\}^{1/2} \\ &\leq C'\varepsilon \|\Phi\|_{1,\varepsilon}. \end{aligned}$$

The contribution of $\bar{\varphi}_2^{(4)}$ can be estimated similarly, and this concludes the proof of the proposition. \square

Remark 3.3. (1) The assumption that $\text{supp } \varphi$ and $\text{supp } \rho$ separate is needed to treat the terms $\bar{\varphi}_2^{(3)}$ and $\bar{\varphi}_2^{(4)}$. For $\bar{\varphi}_2^{(2)}$, this assumption is unnecessary by changing $\frac{1}{12}$ into $\frac{1}{12} + 2J_2^\varepsilon(x) + J_3^\varepsilon(x)$ in the definition of $B^\varepsilon(x, Y)$.

(2) Due to the symmetry of $\eta(x)$, we obtain the constant $\frac{1}{12}$. For general asymmetric η , if it satisfies $\int_0^\infty \eta_4(y) dy = \int_0^\infty \eta_2(y) dy$ (i.e., $P(R_1 + R_2 > 0) = P(R_1 > 0)$ to treat $\bar{\varphi}_2^{(3)}$ and $\bar{\varphi}_2^{(4)}$), this constant $\frac{1}{12}$ is replaced by

$$J_1 = P(R_1 + R_3 > 0, R_2 + R_3 > 0) - P(R_1 > 0)^2,$$

where $\{R_1, R_2, R_3\}$ are independent random variables distributed under $\eta_2(y) dy$. For example, if $\text{supp } \eta \subset [0, \infty)$ (or $(-\infty, 0]$), then $R_1, R_2, R_3 > 0$ a.s. and we have $J_1 = 0$.

3.4 SPDE on \mathbb{S}_M

The arguments developed in Sections 3.1–3.3 work for the SPDE's on $\mathbb{S}_M, M \geq 2$, instead of \mathbb{R} in a similar way. We outline it. Let $h^{\varepsilon, M}(t, x)$, $x \in \mathbb{S}_M$ be the solution of the SPDE (1.12). It is periodically extended on \mathbb{R} . Then, the Cole-Hopf transformed process $Z^{\varepsilon, M}(t, x) = e^{h^{\varepsilon, M}(t, x)}$ satisfies the SPDE (1.13). Taking ρ as in Section 3.2, we consider wrapped processes $g^{\varepsilon, M}(t, x)$ and $Y^{\varepsilon, M}(t, x)$ of $h^{\varepsilon, M}(t, x)$ and $Z^{\varepsilon, M}(t, x)$, respectively. Note that the integral $h^{\varepsilon, M}(t, \rho) = \int_{\mathbb{S}_M} h^{\varepsilon, M}(t, x) \rho(x) dx$ is defined in a periodic sense; that is, $\int_{-M/2}^{M/2} h^{\varepsilon, M}(t, x) \rho(x) dx$ recalling that $M \geq 2$.

We take the initial distribution of $h^{\varepsilon, M}(0, \cdot)$ to be $\pi \otimes \nu^{\varepsilon, M}$ under the map (3.1) with \mathbb{R} replaced by \mathbb{S}_M . Then, the probability measure $\pi \otimes \nu^{\varepsilon, M}$ is invariant under $g^{\varepsilon, M}(t, x)$ as in Lemma 3.1. Lemma 3.2 is also parallel:

Lemma 3.10. $Y^{\varepsilon,M}(t, x)$ satisfies the following equation in generalized functions' sense:

$$(3.27) \quad Y^{\varepsilon,M}(t, x) = Y^{\varepsilon,M}(0, x) + \frac{1}{2} \int_0^t \partial_x^2 Y^{\varepsilon,M}(s, x) ds + \int_0^t A^\varepsilon(x, Y^{\varepsilon,M}(s)) ds \\ + \int_0^t Y^{\varepsilon,M}(s, x) dW^\varepsilon(s, x) + N^{\varepsilon,M}(t, x), \quad x \in \mathbb{S}_M,$$

where $A^\varepsilon(x, Y)$ is defined by (3.5) with convolution $*\eta_2^\varepsilon$ considered in a periodic sense,

$$(3.28) \quad N^{\varepsilon,M}(t, x) = \int_0^t \left\{ (e-1)1_{\{Y_\rho^{\varepsilon,M}(s-) = 1\}} + (e^{-1}-1)1_{\{Y_\rho^{\varepsilon,M}(s-) = e\}} \right\} \\ \times Y^{\varepsilon,M}(s-, x) N^{\varepsilon,M}(ds),$$

and $N^{\varepsilon,M}(t) = -[h^{\varepsilon,M}(t, \rho)]$.

The limit ν of ν^ε as $\varepsilon \downarrow 0$ was identified with the distribution of Gaussian random measure $X = \{X(A); A \in \mathcal{B}(\mathbb{R})\}$. The limit ν^M of $\nu^{\varepsilon,M}$ as $\varepsilon \downarrow 0$ is nothing but $X = \{X(A); A \in \mathcal{B}([0, M])\}$ restricted on $[0, M)$ and conditioned to be $X([0, M]) = 0$. Such conditional random variables $X^M = \{X^M(A); A \in \mathcal{B}([0, M])\}$ can be realized by $X^M((a, b]) = B^M(b) - B^M(a)$, $0 \leq a < b \leq M$, where $B^M = \{B^M(x); x \in [0, M]\}$ is the pinned Brownian motion such that $B^M(0) = B^M(M) = 0$.

The Wiener-Itô expansion (3.8) is modified in the following way under ν^M : Taking a usual standard Brownian motion $B = \{B(x); x \in [0, M]\}$ such that $B(0) = 0$, the pinned Brownian motion is realized as

$$(3.29) \quad B^M(x) = B(x) - \frac{x}{M}B(M), \quad x \in [0, M],$$

so that $B^M(0) = B^M(M) = 0$. By applying Itô's formula and noting $B(M) = \int_0^M dB(x)$, the multiple Wiener integral with respect to B^M can be rewritten into that with respect to B as follows:

$$I^M(\varphi_n) := \frac{1}{n!} \int_{\mathbb{S}_M^n} \varphi_n(x_1, \dots, x_n) dB^M(x_1) \cdots dB^M(x_n) \\ = \frac{1}{n!} \int_{[0, M]^n} \varphi_n^M(x_1, \dots, x_n) dB(x_1) \cdots dB(x_n) = I(\varphi_n^M),$$

where φ_n^M is defined by

$$(3.30) \quad \varphi_n^M(x_1, \dots, x_n) = \sum_{k=0}^n \frac{(-1)^k}{M^k} \sum_{1 \leq i_1 < \dots < i_k \leq n} \int_{[0, M]^k} \check{\varphi}_{n; i_1, \dots, i_k} dy_{i_1} \cdots dy_{i_k}, \\ \check{\varphi}_{n; i_1, \dots, i_k} = \varphi_n(x_1, \dots, x_{i_1-1}, y_{i_1}, x_{i_1+1}, \dots, x_{i_k-1}, y_{i_k}, x_{i_k+1}, \dots, x_n).$$

Note that φ_n^M is symmetric in $\underline{x} = (x_1, \dots, x_n)$ and satisfies

$$(3.31) \quad \int_{[0, M]} \varphi_n^M(x_1, \dots, x_n) dx_i = 0,$$

for every $1 \leq i \leq n$. Thus, $\Phi \in L^2(\tilde{\mathcal{C}}_M, \nu^M)$ has a Wiener-Itô expansion

$$\Phi = \sum_{n=0}^{\infty} I(\varphi_n^M),$$

with some $\varphi_0 \in \mathbb{R}$ and $\varphi_n^M \in \hat{L}_0^2([0, M]^n) := \{\varphi_n^M \in \hat{L}^2([0, M]^n); \varphi_n^M \text{ satisfies the condition (3.31)}\}$. The quotient space $\tilde{\mathcal{C}}_M$ is defined similarly to $\tilde{\mathcal{C}}$.

Note that, for two symmetric functions $f_n = f_n(\underline{x})$ and $\varphi_n = \varphi_n(\underline{x})$, from (3.30), we see that

$$(3.32) \quad \int_{[0, M]^n} f_n^M(\underline{x}) \varphi_n(\underline{x}) d\underline{x} = \int_{[0, M]^n} f_n(\underline{x}) \varphi_n^M(\underline{x}) d\underline{x},$$

and $(\varphi_n^M)^M = \varphi_n^M$, or $\varphi_n^M = \varphi_n$ for $\varphi_n \in \hat{L}_0^2([0, M]^n)$. Moreover, under the representation (3.29) of B^M , (3.16) is modified as

$$\begin{aligned} (\partial_x h)^2 &= \left\{ \int_{[0, M]} \eta^\varepsilon(x - y) dB^M(y) \right\}^2 \\ &= \left\{ \int_{[0, M]} \left(\eta^\varepsilon(x - y) - \frac{1}{M} \right) dB(y) \right\}^2 = \Psi^{\varepsilon, M}(x) + \xi^{\varepsilon, M}, \end{aligned}$$

where $\xi^{\varepsilon, M} = \xi^\varepsilon - \frac{1}{M}$ and

$$\begin{aligned} (3.33) \quad \Psi^{\varepsilon, M}(x) &= \int_{[0, M]^2} \left(\eta^\varepsilon(x - x_1) - \frac{1}{M} \right) \left(\eta^\varepsilon(x - x_2) - \frac{1}{M} \right) dB(x_1) dB(x_2) \\ &= \int_{[0, M]^2} \left(\eta^\varepsilon(x - x_1) \eta^\varepsilon(x - x_2) \right)^M dB(x_1) dB(x_2). \end{aligned}$$

Theorem 3.3 can be reformulated as follows in the present setting:

Theorem 3.11. *For every $\varphi \in C(\mathbb{S}_M)$ satisfying $\text{supp } \varphi \cap \text{supp } \rho = \emptyset$ (so that $\text{dist}(\text{supp } \varphi, \text{supp } \rho) > 0$ in a periodic sense), we have that*

$$\lim_{\varepsilon \downarrow 0} E^{\pi \otimes \nu^{\varepsilon, M}} \left[\sup_{0 \leq t \leq T} \left\{ \int_0^t \hat{A}^\varepsilon(\varphi, Y^{\varepsilon, M}(s)) ds \right\}^2 \right] = 0,$$

where $\hat{A}^\varepsilon(\varphi, Y)$ is defined similarly to that in Theorem 3.3 by taking the integral over \mathbb{S}_M .

Proof. We modify the proofs of Proposition 3.7 and Theorem 3.3. By (3.33), the expectation (3.22) under $\nu^{\varepsilon, M}$ in the present setting is equal to

$$\begin{aligned} (3.34) \quad \int_{[0, M]^2} \bar{\varphi}_2^\varepsilon(x_1, x_2) &\left(\int_{\mathbb{S}_M} \left(\eta^\varepsilon(y - x_1) \eta^\varepsilon(y - x_2) \right)^M \eta_2^\varepsilon(x - y) dy \right. \\ &\left. - \left(\eta^\varepsilon(x - x_1) \eta^\varepsilon(x - x_2) \right)^M \right) dx_1 dx_2, \end{aligned}$$

where $\bar{\varphi}_2^\varepsilon$ is defined from $\bar{\varphi}_2$ given below (3.23) with \mathbb{R}^n replaced by \mathbb{S}_M^n and φ_n, ψ_n replaced by $P^M \varphi_n * (\eta^\varepsilon)^{\otimes n}, P^M \psi_n * (\eta^\varepsilon)^{\otimes n}$, respectively. The operator P^M is defined by $P^M \varphi_n = \varphi_n^M$ and note that the constant $\xi^{\varepsilon, M}$ plays no role in (3.34).

By (3.32), the operator P^M acting on $\psi_n, \psi_{n+1}, \psi_{n+2}$ can be moved to $P^M \varphi_{n+2}, P^M \varphi_{n+1}, P^M \varphi_n$ under the integrals in the variable \underline{u} and to $(\eta^\varepsilon(y-x_1)\eta^\varepsilon(y-x_2))^M$ under the integrals in the variable (x_1, x_2) . Note that we can separate P^M in these two variables. Thus, we can remove P^M from $\psi_n, \psi_{n+1}, \psi_{n+2}$. Writing $P^M \varphi_n, P^M \varphi_{n+1}, P^M \varphi_{n+2}$ simply by $\varphi_n, \varphi_{n+1}, \varphi_{n+2}$ again and noting $(P^M)^2 = P^M$, we can also drop P^M from $\varphi_n, \varphi_{n+1}, \varphi_{n+2}$.

Now, we expand

$$(3.35) \quad \left(\eta^\varepsilon(x-x_1)\eta^\varepsilon(x-x_2) \right)^M = \eta^\varepsilon(x-x_1)\eta^\varepsilon(x-x_2) - \frac{1}{M}(\eta^\varepsilon(x-x_1) + \eta^\varepsilon(x-x_2)) + \frac{1}{M^2}.$$

The contribution to (3.34) from the first term of (3.35) is exactly the same as in the proof of Proposition 3.7. Note that the expectation $\frac{1}{12} E^{\nu^{\varepsilon, M}}[\frac{Y(x)}{Y_\rho} \Phi]$ can be computed similarly in the present setting. The contribution of the last constant $\frac{1}{M^2}$ cancels when we take the difference in (3.34).

The contribution to (3.34) from the second term of (3.35) becomes

$$\begin{aligned} & -\frac{2}{M} \int_{[0, M]^2} \bar{\varphi}_2^\varepsilon(x_1, x_2) \left(\int_{\mathbb{S}_M} \eta^\varepsilon(y-x_1)\eta_2^\varepsilon(x-y)dy - \eta^\varepsilon(x-x_1) \right) dx_1 dx_2 \\ & = -\frac{2}{M} \int_{[0, M]^2} \bar{\varphi}_2^\varepsilon(x_1, x_2) (\eta_3^\varepsilon(x-x_1) - \eta^\varepsilon(x-x_1)) dx_1 dx_2 \end{aligned}$$

by the symmetry of $\bar{\varphi}_2^\varepsilon(x_1, x_2)$. However, since $\int_{[0, M]} \varphi_{n+2}(\underline{u}, x_1, x_2) dx_2 = \int_{[0, M]} \varphi_{n+1}(\underline{u}, x_2) dx_2 = 0$ by (3.31), the contributions of $\bar{\varphi}_2^{(1)}$ and $\bar{\varphi}_2^{(4)}$ vanish. It is therefore enough to compute the contributions of $\bar{\varphi}_2^{(2)}$ and $\bar{\varphi}_2^{(3)}$ only.

The computation of the contribution of

$$\int_{[0, M]^2} \bar{\varphi}_2^{(2), \varepsilon}(x_1, x_2) (\eta_3^\varepsilon(x-x_1) - \eta^\varepsilon(x-x_1)) dx_1 dx_2,$$

with $\bar{\varphi}_2^{(2), \varepsilon}$ defined from $\bar{\varphi}_2^{(2)}$ by replacing φ_n, ψ_{n+2} by $\varphi_n * (\eta^\varepsilon)^{\otimes n}, \psi_{n+2} * (\eta^\varepsilon)^{\otimes(n+2)}$, respectively, can be essentially reduced to the computation of

$$\begin{aligned} & \int_{[0, M]^2} \phi_x^{\otimes 2} * (\eta^\varepsilon)^{\otimes 2}(x_1, x_2) (\eta_3^\varepsilon(x-x_1) - \eta^\varepsilon(x-x_1)) dx_1 dx_2 \\ & = \{\phi_x * \eta_4^\varepsilon(x) - \phi_x * \eta_2^\varepsilon(x)\} \int_{[0, M]} (\phi_x * \eta^\varepsilon)(x_2) dx_2 \\ & = \{\theta * \eta_4^\varepsilon(x) - \theta * \eta_2^\varepsilon(x)\} \int_{[0, M]} (\phi_x * \eta^\varepsilon)(x_2) dx_2. \end{aligned}$$

As we saw in the proof of Proposition 3.7, the support of this function is contained in $(\text{supp } \rho)^{4\varepsilon}$ so that it vanishes when we multiply the test function $\varphi = \varphi(x)$.

The computation of the contribution of

$$\int_{[0,M]^2} \bar{\varphi}_2^{(3),\varepsilon}(x_1, x_2) (\eta_3^\varepsilon(x - x_1) - \eta^\varepsilon(x - x_1)) dx_1 dx_2,$$

with $\bar{\varphi}_2^{(3),\varepsilon}$ defined from $\bar{\varphi}_2^{(3)}$ similarly as above can be essentially reduced to the computation of

$$\begin{aligned} & \int_{\mathbb{S}_M^{n+1}} \phi_x^{\otimes n}(\underline{u})(\phi_x * \eta^\varepsilon)(x_2) d\underline{u} dx_2 \\ & \times \{f_{n+1} * ((\eta^\varepsilon)^{\otimes n} \otimes \eta_3^\varepsilon)(\underline{u}, x) - f_{n+1} * ((\eta^\varepsilon)^{\otimes n} \otimes \eta^\varepsilon)(\underline{u}, x)\}. \end{aligned}$$

However, the difference in the braces can be rewritten as

$$E[f_{n+1}(\underline{u} + U^\varepsilon, x + Y^\varepsilon + X_1^\varepsilon) - f_{n+1}(\underline{u} + U^\varepsilon, x + X_1^\varepsilon)],$$

where $U^\varepsilon = \{U_i^\varepsilon\}_{i=1}^n$, $U_i^\varepsilon \stackrel{\text{law}}{=} \eta^\varepsilon$, $X_1^\varepsilon \stackrel{\text{law}}{=} \eta^\varepsilon$, $Y^\varepsilon \stackrel{\text{law}}{=} \eta_2^\varepsilon$ and $\{U_i^\varepsilon, X_1^\varepsilon, Y^\varepsilon\}$ are independent. This can be estimated exactly in the same way as $I_n^{(3)}$ in the proof of Proposition 3.7. Therefore, we have a parallel assertion to Proposition 3.7 in the present setting and this completes the proof of the theorem. \square

The proof of Theorems 3.3 or 3.11 can be extended to obtain the following lemma for $h^{\varepsilon,M}(t, x)$. To prove this lemma, we need Poincaré inequality so that this holds only on \mathbb{S}_M and not on \mathbb{R} .

Lemma 3.12. *Assume that a measurable and bounded function $\varphi = \varphi(s, x)$ on $[0, T] \times \mathbb{S}_M$ is given. Then, we have*

$$(3.36) \quad \begin{aligned} & \sup_{0 < \varepsilon < 1} E^{\pi \otimes \nu^{\varepsilon,M}} \left[\sup_{0 \leq t \leq T} \left(\int_0^t H^{\varepsilon,M}(\varphi(s, \cdot), \partial h^{\varepsilon,M}(s)) ds \right)^2 \right] \\ & \leq 12M^2 \int_0^T \|\varphi(s, \cdot)\|_{L^\infty(\mathbb{S}_M)}^2 ds, \end{aligned}$$

where

$$(3.37) \quad H^{\varepsilon,M}(x, \partial h) = \frac{1}{2}((\partial_x h)^2 - \xi^{\varepsilon,M}) * \eta_2^\varepsilon(x)$$

and $H^{\varepsilon,M}(\varphi, \partial h) = \int_{\mathbb{S}_M} H^{\varepsilon,M}(x, \partial h) \varphi(x) dx$.

Proof. By Lemma 3.6, noting that $H^{\varepsilon,M}(\varphi, \partial h)$ is a tilt variable, the expectation in (3.36) is bounded by

$$(3.38) \quad 24 \int_0^T \sup_{\Phi \in L^2(\nu^{\varepsilon,M})} \left\{ 2E^{\nu^{\varepsilon,M}} \left[\tilde{\Psi}^{\varepsilon,M}(\varphi(s, \cdot)) \Phi \right] - \|\Phi\|_{1,\varepsilon}^2 \right\} ds,$$

where $\tilde{\Psi}^{\varepsilon,M}(\varphi) = \frac{1}{2} \int_{\mathbb{S}_M} \Psi^{\varepsilon,M} * \eta_2^\varepsilon(x) \varphi(x) dx$, $\Psi^{\varepsilon,M}(x)$ is given by (3.33) and $\|\Phi\|_{1,\varepsilon}^2$ is defined on \mathbb{S}_M . Since $\Psi^{\varepsilon,M}$ is a second order Wiener chaos, we have that

$$2E^{\nu^{\varepsilon,M}} \left[\tilde{\Psi}^{\varepsilon,M}(\varphi) \Phi \right] = \int_{\mathbb{S}_M} \varphi * \eta_2^\varepsilon(x) dx \int_{\mathbb{S}_M^2} f_2(x_1, x_2) \left(\eta^\varepsilon(x - x_1) \eta^\varepsilon(x - x_2) \right)^M dx_1 dx_2$$

$$= \int_{\mathbb{S}_M^2} f_2^M(x_1, x_2) \psi(x_1, x_2) dx_1 dx_2,$$

where

$$\psi(x_1, x_2) = \int_{\mathbb{S}_M} \varphi * \eta_2^\varepsilon(x) \eta^\varepsilon(x - x_1) \eta^\varepsilon(x - x_2) dx,$$

and $f_2 = \varphi_2 \otimes (\eta^\varepsilon)^{\otimes 2}$ with the kernel $\varphi_2 \in \hat{L}_0^2([0, M]^2)$ of the second order Wiener chaos of Φ . Note that (3.31) implies $\int_{\mathbb{S}_M} \varphi_2(x_1, x_2) dx_i = 0$, $i = 1, 2$, so that

$$(3.39) \quad \int_{\mathbb{S}_M} f_2(x_1, x_2) dx_i = 0.$$

This shows $f_2^M = f_2$. Moreover, Lemma 3.5 (similar on \mathbb{S}_M) implies

$$(3.40) \quad \frac{1}{2} \int_{\mathbb{S}_M^2} \left(\frac{\partial f_2}{\partial x_1}(x_1, x_2) \right)^2 dx_1 dx_2 \leq \|\Phi\|_{1,\varepsilon}^2.$$

We now estimate

$$(3.41) \quad \left| 2E^{\nu^{\varepsilon,M}} \left[\tilde{\Psi}^{\varepsilon,M}(\varphi) \Phi \right] \right| \leq \int_{\mathbb{S}_M} \|f_2(\cdot, x_2)\|_{L^\infty(\mathbb{S}_M)} \|\psi(\cdot, x_2)\|_{L^1(\mathbb{S}_M)} dx_2.$$

However, we easily see that

$$\|\psi(\cdot, x_2)\|_{L^1(\mathbb{S}_M)} \leq \|\varphi\|_{L^\infty(\mathbb{S}_M)},$$

for every $x_2 \in \mathbb{S}_M$ and, by (3.39) with $i = 1$,

$$\begin{aligned} |f_2(x_1, x_2)| &= \left| f_2(x_1, x_2) - \frac{1}{M} \int_{\mathbb{S}_M} f_2(y, x_2) dy \right| \\ &= \frac{1}{M} \left| \int_{\mathbb{S}_M} dy \int_y^{x_1} \frac{\partial f_2}{\partial z}(z, x_2) dz \right| \leq \int_{\mathbb{S}_M} \left| \frac{\partial f_2}{\partial z}(z, x_2) \right| dz. \end{aligned}$$

Thus, by (3.41), Schwarz's inequality and (3.40), we obtain

$$\begin{aligned} \left| 2E^{\nu^{\varepsilon,M}} \left[\tilde{\Psi}^{\varepsilon,M}(\varphi) \Phi \right] \right| &\leq \|\varphi\|_{L^\infty(\mathbb{S}_M)} \int_{\mathbb{S}_M^2} \left| \frac{\partial f_2}{\partial x_1}(x_1, x_2) \right| dx_1 dx_2 \\ &\leq M \|\varphi\|_{L^\infty(\mathbb{S}_M)} \left(\int_{\mathbb{S}_M^2} \left| \frac{\partial f_2}{\partial x_1}(x_1, x_2) \right|^2 dx_1 dx_2 \right)^{1/2} \\ &\leq \sqrt{2} M \|\varphi\|_{L^\infty(\mathbb{S}_M)} \|\Phi\|_{1,\varepsilon}. \end{aligned}$$

Combining this with (3.38), the conclusion is shown similarly to the last part of the proof of Theorem 3.3. \square

Remark 3.4. (1) For $p > 1$, L^p -norms of ψ diverge as $\varepsilon \downarrow 0$ in general.

(2) On \mathbb{R} , we have the same estimate as (3.41) and, if $\text{supp } \varphi \subset [-K, K]$, then $\|\psi(\cdot, x_2)\|_{L^1(\mathbb{R})} =$

0 if $|x_2| \geq K+3\varepsilon$. Therefore, Morrey's theorem ([1], p.97 (8)), which implies $\|f_2(\cdot, x_2)\|_{L^\infty(\mathbb{R})} \leq C\|f_2(\cdot, x_2)\|_{H^1(\mathbb{R})}$, shows that

$$\left| 2E^{\nu^\varepsilon} \left[\tilde{\Psi}^\varepsilon(\varphi)\Phi \right] \right| \leq C\sqrt{K}\|\varphi\|_{L^\infty(\mathbb{R})} \{ \|\partial f_2/\partial x_1\|_{L^2(\mathbb{R}^2)} + \|f\|_{L^2(\mathbb{R}^2)} \}.$$

However, since Poincaré inequality is missing, this cannot be bounded by $\|\Phi\|_{1,\varepsilon}$, because of the last term $\|f\|_{L^2(\mathbb{R}^2)}$. This estimate can be easily extended to φ such that $\sup_{x \in \mathbb{R}} |\varphi(x)|\sqrt{1+|x|} < \infty$.

This lemma is applied to prove the following proposition, which shows that the height average cannot move very quickly. Recall that the initial distribution of $h^{\varepsilon,M}(t, x)$ is given by $h^{\varepsilon,M}(0, \cdot) \stackrel{\text{law}}{=} \pi \otimes \nu^{\varepsilon,M}$ under the map (3.1) defined on \mathbb{S}_M . Note that the wrapping is not introduced for $h^{\varepsilon,M}(t, x)$. This result will be used to remove the wrapping from $Y^{\varepsilon,M}(t, x)$ in the next section.

Proposition 3.13. *For every $T > 0$ and $\varphi \in C^2(\mathbb{S}_M)$,*

$$(3.42) \quad \sup_{0 < \varepsilon < 1} E \left[\sup_{0 \leq t \leq T} h^{\varepsilon,M}(t, \varphi)^2 \right] < \infty.$$

Proof. For every $\varphi \in C^2(\mathbb{S}_M)$, from the SPDE (1.12), we have

$$(3.43) \quad \begin{aligned} h^{\varepsilon,M}(t, \varphi) = & h^{\varepsilon,M}(0, \varphi) + \frac{1}{2} \int_0^t h^{\varepsilon,M}(s, \varphi'') ds + \int_0^t H^{\varepsilon,M}(\varphi, \partial h^{\varepsilon,M}(s)) ds \\ & + \frac{1}{2} (\xi^{\varepsilon,M} - \xi^\varepsilon) t \langle \varphi, 1 \rangle_{\mathbb{S}_M} + \langle W(t), \varphi * \eta^\varepsilon \rangle_{\mathbb{S}_M}. \end{aligned}$$

By Lemma 3.12, it is shown that

$$\sup_{0 < \varepsilon < 1} E \left[\sup_{0 \leq t \leq T} \left(\int_0^t H^{\varepsilon,M}(\varphi, \partial h^{\varepsilon,M}(s)) ds \right)^2 \right] < \infty.$$

It is easy to see that $\xi^{\varepsilon,M} - \xi^\varepsilon = -\frac{1}{M}$ is finite,

$$E \left[\sup_{0 \leq t \leq T} \langle W(t), \varphi * \eta^\varepsilon \rangle_{\mathbb{S}_M}^2 \right] \leq 4E[\langle W(T), \varphi * \eta^\varepsilon \rangle_{\mathbb{S}_M}^2] = 4T \|\varphi * \eta^\varepsilon\|_{L^2(\mathbb{S}_M)}^2 \leq 4T \|\varphi\|_{L^2(\mathbb{S}_M)}^2 < \infty,$$

and

$$E \left[\sup_{0 \leq t \leq T} \left(\int_0^t h^{\varepsilon,M}(s, \varphi'') ds \right)^2 \right] \leq T \int_0^T E[h^{\varepsilon,M}(s, \varphi'')^2] ds.$$

Therefore, once we can prove

$$(3.44) \quad \sup_{0 < \varepsilon < 1} \sup_{0 \leq t \leq T} E[h^{\varepsilon,M}(s, \varphi)^2] < \infty,$$

for every $\varphi \in C(\mathbb{S}_M)$, (3.43) shows (3.42).

The next task is to give the proof of (3.44). To this end, we rewrite the equation for $h^{\varepsilon,M}(t, x)$ into a mild form:

$$\begin{aligned} h^{\varepsilon,M}(t, \varphi) &= \int_{\mathbb{S}_M} h^{\varepsilon,M}(0, y) \psi(t, y) dy + \int_0^t \int_{\mathbb{S}_M} \psi(t-s, y) H^{\varepsilon,M}(y, \partial h^{\varepsilon,M}(s)) ds dy \\ &\quad + \frac{1}{2} (\xi^{\varepsilon,M} - \xi^\varepsilon) \int_0^t \int_{\mathbb{S}_M} \psi(t-s, y) ds dy + \int_0^t \int_{\mathbb{S}_M} \psi(t-s, y) W^\varepsilon(ds dy), \end{aligned}$$

where $\psi(t, y) = \int_{\mathbb{S}_M} p^M(t, x, y) \varphi(x) dx$ and p^M is the heat kernel on \mathbb{S}_M . We apply Lemma 3.12 by dropping $\sup_{0 \leq t \leq T}$ and then regarding $\varphi(s, y) = \psi(t-s, y)$ for fixed t , we have

$$\begin{aligned} &E^{\pi \otimes \nu^{\varepsilon,M}} \left[\left(\int_0^t \int_{\mathbb{S}_M} \psi(t-s, y) H^{\varepsilon,M}(y, \partial h^{\varepsilon,M}(s)) ds dy \right)^2 \right] \\ &\leq 12M^2 \int_0^t \|\psi(t-s, \cdot)\|_{L^\infty(\mathbb{S}_M)}^2 ds \leq 12M^2 t \|\varphi\|_{L^\infty(\mathbb{S}_M)}^2 < \infty. \end{aligned}$$

We also see

$$E^{\pi \otimes \nu^{\varepsilon,M}} \left[\left(\int_0^t \int_{\mathbb{S}_M} \psi(t-s, y) W^\varepsilon(ds dy) \right)^2 \right] = \int_0^t \int_{\mathbb{S}_M} (\psi(t-s, \cdot) * \eta^\varepsilon(y))^2 ds dy \leq C < \infty.$$

Since the first term has a uniform bound recalling $h^{\varepsilon,M}(0, \cdot) \stackrel{\text{law}}{=} \pi \otimes \nu^{\varepsilon,M}$ and the integral in the third term is simply $t \langle \varphi, 1 \rangle_{\mathbb{S}_M}$, this completes the proof of (3.44). \square

Remark 3.5. As we will see in the next subsection, $N^{\varepsilon,M}(t, \varphi) = \int_{\mathbb{S}_M} N^{\varepsilon,M}(t, x) \varphi(x) dx$ in (3.27) converges weakly in $L^2(\Omega)$ as $\varepsilon \downarrow 0$, since all other terms do. Therefore, its L^2 -norm is automatically bounded in ε :

$$(3.45) \quad \sup_{0 < \varepsilon < 1} E[(N^{\varepsilon,M}(t, \varphi))^2] < \infty.$$

Proposition 3.13 gives a stronger estimate with supremum in t inside the expectation.

3.5 Proof of Theorem 1.1

We are now at the position to complete the proof of Theorem 1.1. We fix $M \geq 2$ (or $M \geq 1$ by making the support of ρ smaller) and denote $h^{\varepsilon,M}(t, x)$, $Z^{\varepsilon,M}(t, x)$, $Y^{\varepsilon,M}(t, x)$ simply by $h^\varepsilon(t, x)$, $Z^\varepsilon(t, x)$, $Y^\varepsilon(t, x)$, respectively, in this subsection.

Step 1. Let $h^\varepsilon(t, x)$ be the solution of the SPDE (1.12) such that $h^\varepsilon(0, 0) = h_0 \in \mathbb{R}$ and $\nabla h^\varepsilon(0) \stackrel{\text{law}}{=} \nu^{\varepsilon,M}$. We may assume $h_0 = 0$ without loss of generality. Then, $h^\varepsilon(0, \cdot) (\in \mathcal{C}_M) \stackrel{\text{law}}{=} \delta_0 \otimes \nu^{\varepsilon,M}$, $0 < \varepsilon < 1$ is tight.

Lemma 3.14. (1) The uniform bound (3.42) holds also for this $h^\varepsilon(t, x)$.
(2) The tilt variable of $h^\varepsilon(t, \cdot)$ under the map (3.1) on \mathbb{S}_M is $\nu^{\varepsilon,M}$ -distributed for all $t \geq 0$.

Proof. The bound (3.42) was shown for the solution $\tilde{h}^\varepsilon(t, x)$ of the SPDE (1.12) such that $\tilde{h}^\varepsilon(0, \cdot) = (\tilde{h}^\varepsilon(0, \rho), \tilde{h}^\varepsilon(0, \cdot) - \tilde{h}^\varepsilon(0, \rho)) \stackrel{\text{law}}{=} \pi \otimes \nu^{\varepsilon, M}$ under the map (3.1) on \mathbb{S}_M . However, two solutions have a simple relation: $\tilde{h}^\varepsilon(t, x) = h^\varepsilon(t, x) - h^\varepsilon(0, \rho) + m$ with a π -distributed random variable m (which is independent of tilt variables of $h^\varepsilon(0, \cdot)$). Therefore, we see that $|h^\varepsilon(t, \varphi) - \tilde{h}^\varepsilon(t, \varphi)| \leq \{1 + |h^\varepsilon(0, \rho)|\} \int_{\mathbb{S}_M} |\varphi(x)| dx$ so that (3.42) holds for the solution $h^\varepsilon(t, x)$ we are now considering. The second assertion (2) follows by noting that the tilt variables of $h^\varepsilon(t, \cdot)$ and $\tilde{h}^\varepsilon(t, \cdot)$ coincide: $h^\varepsilon(t, x) - h^\varepsilon(t, \rho) = \tilde{h}^\varepsilon(t, x) - \tilde{h}^\varepsilon(t, \rho)$. \square

Lemma 3.14 taking $\varphi = \rho$ in (3.42) shows that, for every fixed $t > 0$, $h^\varepsilon(t, \cdot)$, $0 < \varepsilon < 1$, is tight on \mathcal{C}_M . We fix $T > 0$ arbitrarily. Then, from Lemma 3.14 again, $(h^\varepsilon(0, \cdot), h^\varepsilon(T, \cdot), X^\varepsilon := \sup_{0 \leq t \leq T} |h^\varepsilon(t, \rho)|, W(t))$, $0 < \varepsilon < 1$, is jointly tight on $\mathcal{C}_M \times \mathcal{C}_M \times \mathbb{R} \times C([0, T], H^{-\alpha}(\mathbb{S}_M))$ with suitably chosen $\alpha > 0$. In particular, every subsequence $\{\varepsilon' \downarrow 0\}$ of $\{\varepsilon \in (0, 1)\}$ contains a weakly convergent subsequence (in law sense) $(h^{\varepsilon''}(0, \cdot), h^{\varepsilon''}(T, \cdot), X^{\varepsilon''}, W(t))$. Thus, by Skorohod's theorem, we can find a probability space $(\tilde{\Omega}, \tilde{P})$ and $(\tilde{h}^{\varepsilon''}(0, \cdot), \tilde{h}^{\varepsilon''}(T, \cdot), \tilde{X}^{\varepsilon''}, \tilde{W}_{\varepsilon''}(t))$ defined on this space having the same law as $(h^{\varepsilon''}(0, \cdot), h^{\varepsilon''}(T, \cdot), X^{\varepsilon''}, W(t))$, and $(\tilde{h}(0, \cdot), \tilde{h}(T, \cdot), \tilde{X}, \tilde{W}(t))$ also defined on this space such that $(\tilde{h}^{\varepsilon''}(0, \cdot), \tilde{h}^{\varepsilon''}(T, \cdot), \tilde{X}^{\varepsilon''}, \tilde{W}_{\varepsilon''}(t))$ converges to $(\tilde{h}(0, \cdot), \tilde{h}(T, \cdot), \tilde{X}, \tilde{W}(t))$ \tilde{P} -a.s. $\tilde{\omega}$ as $\varepsilon'' \downarrow 0$.

From the space-time Gaussian white noise $\tilde{W}_{\varepsilon''}(t)$, one can construct a smeared noise $\tilde{W}_{\varepsilon''}^{\varepsilon''}(t) := \tilde{W}_{\varepsilon''}(t) * \eta^{\varepsilon''}$ and consider the SPDE (1.13) with initial value $\tilde{Z}^{\varepsilon''}(0, \cdot) = e^{\tilde{h}^{\varepsilon''}(0, \cdot)}$. Then, we have its solution $\tilde{Z}^{\varepsilon''}(t, \cdot)$. $\tilde{Z}^{\varepsilon''}(T, \cdot)$ is consistent with $\tilde{h}^{\varepsilon''}(T, \cdot)$ and $\tilde{Z}^{\varepsilon''}(t, 0)$ is consistent with $\tilde{h}^{\varepsilon''}(t, 0)$, respectively, \tilde{P} -a.s.

For every $L \geq 1$, we define the wrapped process $\tilde{Y}_L^{\varepsilon''}(t, x) \equiv \tilde{Y}_L^{\varepsilon''}(t, x; \tilde{\omega}, m)$ on the extended probability space $(\tilde{\Omega} \times [-L, L], \tilde{P} \otimes \pi_L)$ in such a manner that $\tilde{Y}_L^{\varepsilon''}(t, x) = e^m \tilde{Z}^{\varepsilon''}(t, x)$ modulo $2L$ averaged by ρ in an exponential sense such as (3.3), i.e., $\log(\tilde{Y}_L^{\varepsilon''}(t))_\rho \in [e^{-L}, e^L]$ (instead of $[1, e]$) and $\log(\tilde{Y}_L^{\varepsilon''}(t))_\rho = m + \log(\tilde{Z}^{\varepsilon''}(t))_\rho$ modulo $2L$ with π_L -valued random variable m , where π_L is a uniform probability measure on $[-L, L]$. In other words, the initial value $\log(\tilde{Y}_L^{\varepsilon''})_\rho(0)$ is distributed under π_L .

Step 2. Since $\tilde{X}^{\varepsilon''} = \sup_{0 \leq t \leq T} |\tilde{h}^{\varepsilon''}(t, \rho)|$ converges to \tilde{X} as $\varepsilon'' \downarrow 0$ in \mathbb{R} , \tilde{P} -a.s., and $0 \leq \tilde{X} < \infty$, \tilde{P} -a.s., we see that $\lim_{L \rightarrow \infty} \tilde{P}(\tilde{\Omega}_L) = 1$ holds, where

$$\tilde{\Omega}_L := \left\{ \sup_{0 < \varepsilon'' < 1} \sup_{0 \leq t \leq T} |\tilde{h}^{\varepsilon''}(t, \rho)| \leq \frac{L}{2} \right\} (\subset \tilde{\Omega}).$$

In particular, we have that

$$(3.46) \quad \tilde{Y}_L^{\varepsilon''}(t, x; \tilde{\omega}, m) = e^m \tilde{Z}^{\varepsilon''}(t, x; \tilde{\omega})$$

for all $t \in [0, T]$, $m : |m| \leq \frac{L}{2}$, $0 < \varepsilon'' < 1$ and $\tilde{\omega} \in \tilde{\Omega}_L$.

We prepare the following uniform bound on $\tilde{Y}_L^{\varepsilon''}(t, x)$ defined on $\tilde{\Omega}$.

Lemma 3.15. *We have that*

$$(3.47) \quad \tilde{E}[\tilde{Y}_L^{\varepsilon''}(t, x; \cdot, m)^{2p}] \leq e^{4p^2(|x|+c)+2Lp},$$

for every $0 < \varepsilon'' < 1$, $t \geq 0$, $p \geq 1$, $x \in [-\frac{M}{2}, \frac{M}{2}]$ and $m \in [-L, L]$ with some $c > 0$.

Proof. Since the tilt variable for $\tilde{Y}^{\varepsilon''}(t)$ is distributed under $\nu^{\varepsilon'',M}$, we have that

$$\begin{aligned}
\tilde{E}[\tilde{Y}_L^{\varepsilon''}(t, x)^{2p}] &\leq \tilde{E}[(\tilde{Y}_L^{\varepsilon''}(t))_\rho^{4p}]^{1/2} \tilde{E}\left[\frac{\tilde{Y}_L^{\varepsilon''}(t, x)^{4p}}{(\tilde{Y}_L^{\varepsilon''}(t))_\rho^{4p}}\right]^{1/2} \\
&\leq e^{2Lp} E[e^{4p(B^M(\cdot) - \int_{\mathbb{S}_M} B^M(y)\rho(y)dy) * \eta^{\varepsilon''}(x)}]^{1/2} \\
&= e^{2Lp} E[e^{4p \int_{\mathbb{S}_M} \phi \cdot (u) * \eta^{\varepsilon''}(x) dB^M(u)}]^{1/2} \\
&= e^{2Lp} e^{\frac{1}{4}(4p)^2 \int_{\mathbb{S}_M} (\phi \cdot (u) * \eta^{\varepsilon''}(x) - \frac{1}{M} \int_{\mathbb{S}_M} \phi \cdot (v) * \eta^{\varepsilon''}(x) dv)^2 du} \\
&\leq e^{2Lp} e^{4p^2 \int_{\mathbb{S}_M} (\phi \cdot (u) * \eta^{\varepsilon''}(x))^2 du} \\
&\leq e^{4p^2(|x|+c)+2Lp},
\end{aligned}$$

where $\phi_x(u)$ is defined in (3.20). Note that $\phi \cdot (u) * \eta^{\varepsilon''}(x) = 1_{[u, \infty)} * \eta^\varepsilon(x) + \theta(u)$, $1_{[u, \infty)} * \eta^\varepsilon(x) = 0$ ($u \geq x + \varepsilon$), $1(u \leq x - \varepsilon)$, $\in [0, 1]$ (otherwise), and $\theta(u) = 0(u \geq K)$, $\theta(u) = -1(u \leq -K)$, where $K > 0$ is taken such that $\text{supp } \rho \subset [-K, K] \subset [-1, 1]$. Recall that $\text{supp } \eta^\varepsilon \subset [-\varepsilon, \varepsilon]$. This shows (3.47). \square

Since (3.47) with $p = 1$ implies the weak compactness of $\tilde{Y}_L^{\varepsilon''}(\cdot, \cdot)$, by the similar argument to Krylov and Rozovskii [18], p.1264 and by the diagonal argument in $L \in \mathbb{N}$, one can find a subsequence $\varepsilon''' \downarrow 0$ of ε'' such that

$$(3.48) \quad \tilde{Y}_L^{\varepsilon'''}(t, x) \rightarrow \tilde{Y}_L(t, x; \tilde{\omega}, m),$$

weakly in $L^2([0, T] \times \tilde{\Omega} \times [-L, L], \bar{\mathcal{T}}_L, dtd\tilde{P}d\pi_L; L^2(\mathbb{S}_M))$ with some $\tilde{Y}_L(t, x; \tilde{\omega}, m)$, where $\bar{\mathcal{T}}_L$ is the completion of the σ -field of progressively measurable sets on $[0, T] \times \tilde{\Omega} \times [-L, L]$ with respect to $dtd\tilde{P}d\pi_L$ for every $L \in \mathbb{N}$. This combined with (3.46) shows

$$(3.49) \quad \tilde{Z}^{\varepsilon'''}(t, x) \rightarrow \tilde{Z}(t, x),$$

weakly in $L^2([0, T] \times \tilde{\Omega}_L, \bar{\mathcal{T}}, dtd\tilde{P}; L^2(\mathbb{S}_M))$, for all $L \in \mathbb{N}$ with some $\tilde{Z}(t, x)$, where $\bar{\mathcal{T}}$ is defined similarly to $\bar{\mathcal{T}}_L$. Furthermore, by definition, we see that

$$(3.50) \quad \tilde{Y}_L(t, x; \tilde{\omega}, m) = e^m \tilde{Z}(t, x; \tilde{\omega})$$

in $L^2([0, T], L^2(\mathbb{S}_M))$ for a.e. $m : |m| \leq \frac{L}{2}$, $L \in \mathbb{N}$ and \tilde{P} -a.s. $\tilde{\omega} \in \tilde{\Omega}_L$.

The identity (3.50) can be rewritten as

$$\tilde{Z}(t, x; \tilde{\omega}) = e^{-m} \tilde{Y}_L(t, x; \tilde{\omega}, m),$$

for a.e. $m : |m| \leq \frac{L}{2}$, $L \in \mathbb{N}$ and \tilde{P} -a.s. $\tilde{\omega} \in \tilde{\Omega}_L$.

Step 3. From (3.27) considered on $\tilde{\Omega}$ and modulo $2L$ rather than modulo 1, under a multiplication of $\varphi = \varphi(x) \in C^\infty(\mathbb{S}_M)$ and dropping superscripts M as above, we have that

$$(3.51) \quad \langle \tilde{Y}_L^{\varepsilon'''}(t), \varphi \rangle_{\mathbb{S}_M} = \langle \tilde{Y}_L^{\varepsilon'''}(0), \varphi \rangle_{\mathbb{S}_M}$$

$$\begin{aligned}
& + \int_0^t \langle \tilde{Y}_L^{\varepsilon'''}(s), \frac{1}{2} \partial_x^2 \varphi + \frac{1}{24} \varphi \rangle_{\mathbb{S}_M} ds + \int_0^t \int_{\mathbb{S}_M} \hat{A}^{\varepsilon'''}(x, \tilde{Y}_L^{\varepsilon'''}(s)) \varphi(x) ds dx \\
& + \int_0^t \int_{\mathbb{S}_M} \tilde{Y}_L^{\varepsilon'''}(s, x) \varphi(x) \tilde{W}^{\varepsilon'''}(ds dx) + \int_{\mathbb{S}_M} \tilde{N}_L^{\varepsilon'''}(t, x) \varphi(x) dx,
\end{aligned}$$

where $\tilde{W}^{\varepsilon'''} := \tilde{W}_{\varepsilon'''} * \eta^{\varepsilon'''}$ and $\tilde{N}_L^{\varepsilon'''}$ is defined correspondingly.

By Theorem 3.11, for each fixed $L \in \mathbb{N}$, the third term in the right hand side of (3.51) converges to 0 strongly in $L^2(\tilde{\Omega} \times [-L, L], d\tilde{P} d\pi_L)$ as $\varepsilon(= \varepsilon''') \downarrow 0$, if $\text{supp } \varphi \cap \text{supp } \rho = \emptyset$. The fourth term in the right hand side of (3.51) involving stochastic integrals converges in the following sense:

Lemma 3.16. *For every $0 \leq t \leq T$, as $\varepsilon(= \varepsilon''') \downarrow 0$,*

$$(3.52) \quad \int_0^t \int_{\mathbb{S}_M} \tilde{Y}_L^{\varepsilon'''}(s, x) \varphi(x) \tilde{W}^{\varepsilon'''}(ds dx) \rightarrow \int_0^t \int_{\mathbb{S}_M} \tilde{Y}_L(s, x) \varphi(x) \tilde{W}(ds dx),$$

weakly in $L^2(\tilde{\Omega} \times [-L, L], \tilde{\mathcal{F}}_t, \tilde{P} \otimes \pi_L)$, where $\tilde{\mathcal{F}}_t$ is the σ -field generated by $\{\tilde{W}(s); 0 \leq s \leq t\}$.

Proof. Rewriting as

$$\begin{aligned}
\int_0^t \int_{\mathbb{S}_M} \tilde{Y}_L^{\varepsilon'''}(s, x) \varphi(x) \tilde{W}^{\varepsilon'''}(ds dx) &= \int_0^t \int_{\mathbb{S}_M} \left(\tilde{Y}_L^{\varepsilon'''}(s, \cdot) \varphi(\cdot) \right) * \eta^{\varepsilon'''}(x) \tilde{W}_{\varepsilon'''}(ds dx) \\
&= \int_0^t \int_{\mathbb{S}_M^2} \tilde{Y}_L^{\varepsilon'''}(s, y) \varphi(y) \eta^{\varepsilon'''}(x - y) \tilde{W}_{\varepsilon'''}(ds dx) dy,
\end{aligned}$$

the difference of the both sides of (3.52) is given by

$$\begin{aligned}
& \int_0^t \int_{\mathbb{S}_M^2} \tilde{Y}_L^{\varepsilon'''}(s, y) \varphi(y) \eta^{\varepsilon'''}(x - y) \{ \tilde{W}_{\varepsilon'''}(ds dx) - \tilde{W}(ds dx) \} dy \\
& + \int_0^t \int_{\mathbb{S}_M^2} \{ \tilde{Y}_L^{\varepsilon'''}(s, y) - \tilde{Y}_L(s, y) \} \varphi(y) \eta^{\varepsilon'''}(x - y) \tilde{W}(ds dx) dy \\
& + \int_0^t \int_{\mathbb{S}_M^2} \{ \tilde{Y}_L(s, y) - \tilde{Y}_L(s, x) \} \varphi(y) \eta^{\varepsilon'''}(x - y) \tilde{W}(ds dx) dy \\
& + \int_0^t \int_{\mathbb{S}_M} \tilde{Y}_L(s, x) \{ \varphi * \eta^{\varepsilon'''}(x) - \varphi(x) \} \tilde{W}(ds dx) \\
& =: I^{(1, \varepsilon''')} + I^{(2, \varepsilon''')} + I^{(3, \varepsilon''')} + I^{(4, \varepsilon''')}.
\end{aligned}$$

For the first term $I^{(1, \varepsilon''')}$, since both $\tilde{W}_{\varepsilon'''}$ and \tilde{W} are the space-time Gaussian white noises and $\tilde{W}_{\varepsilon'''}$ converges to \tilde{W} a.s., we see that $\tilde{W}_{\varepsilon'''}$ is also a space-time Gaussian white noise multiplied by a constant $c_{\varepsilon'''}$ converging to 0 as $\varepsilon''' \downarrow 0$. Thus,

$$\tilde{E}[\{I^{(1, \varepsilon''')}\}^2] = c_{\varepsilon'''}^2 \int_0^t \int_{\mathbb{S}_M} \tilde{E} \left[\left(\int_{\mathbb{S}_M} \tilde{Y}_L^{\varepsilon'''}(s, y) \varphi(y) \eta^{\varepsilon'''}(x - y) dy \right)^2 \right] ds dx$$

$$\leq c_{\varepsilon'''}^2 \int_0^t \int_{\mathbb{S}_M} \varphi(y)^2 \tilde{E}[\tilde{Y}_L^{\varepsilon'''}(s, y)^2] ds dy,$$

which converges to 0 as $\varepsilon''' \downarrow 0$ by noting (3.47). For the last term, since $\|\varphi * \eta^{\varepsilon'''} - \varphi\|_{L^\infty(\mathbb{S}_M)} \rightarrow 0$, we easily see that

$$\tilde{E}[\{I^{(4, \varepsilon''')}\}^2] \rightarrow 0 \quad \text{as } \varepsilon''' \downarrow 0.$$

For other terms, note that functions Φ of the forms

$$\Phi = \int_0^t \int_{\mathbb{S}_M} \varphi(s, x, \tilde{\omega}) \tilde{W}(ds dx)$$

with bounded and continuous kernels $\varphi(s, x, \tilde{\omega})$ form a dense family in $L^2(\tilde{\Omega}, \tilde{\mathcal{F}}_t, \tilde{P}) \ominus \{\text{const}\}$. Then, we have that

$$\tilde{E}[I^{(2, \varepsilon''')}\Phi] = \int_0^t ds \int_{\mathbb{S}_M^2} \tilde{E}[\{\tilde{Y}_L^{\varepsilon'''}(s, y) - \tilde{Y}_L(s, y)\}\varphi(s, x, \tilde{\omega})]\varphi(y)\eta^{\varepsilon'''}(x - y) dx dy.$$

If we can replace $\varphi(s, x, \tilde{\omega})$ with $\varphi(s, y, \tilde{\omega})$, then $\eta^{\varepsilon'''}(x - y)$ disappears under the integration in x and this expectation converges to 0 by (3.48) and applying Lebesgue's convergence theorem noting (3.47). The replacement of $\varphi(s, x, \omega)$ with $\varphi(s, y, \omega)$ with a small error negligible as $\varepsilon''' \downarrow 0$ can be justified by the continuity of $\varphi(\cdot, \cdot, \omega)$ noting (3.47).

Similarly for $I^{(3, \varepsilon''')}$, we have that

$$\tilde{E}[I^{(3, \varepsilon''')}\Phi] = \int_0^t ds \int_{\mathbb{S}_M^2} \tilde{E}[\{\tilde{Y}_L(s, y) - \tilde{Y}_L(s, x)\}\varphi(s, x, \tilde{\omega})]\varphi(y)\eta^{\varepsilon'''}(x - y) dx dy.$$

First, by the continuity of $\varphi(\cdot, \cdot, \tilde{\omega})$ and noting (3.47), which implies a corresponding bound on $\tilde{Y}_L(s, y)$, we can replace this by

$$(3.53) \quad \int_0^t ds \int_{\mathbb{S}_M^2} \tilde{E}[\tilde{Y}_L(s, y)\varphi(s, y, \tilde{\omega}) - \tilde{Y}_L(s, x)\varphi(s, x, \tilde{\omega})]\varphi(y)\eta^{\varepsilon'''}(x - y) dx dy$$

with a small error negligible as $\varepsilon''' \downarrow 0$. Then, noting the symmetry of $\eta^{\varepsilon'''}$, we can rewrite (3.53) as

$$\int_0^t ds \int_{\mathbb{S}_M^2} \tilde{E}[\tilde{Y}_L(s, y)\varphi(s, y, \tilde{\omega})](\varphi(y) - \varphi(x))\eta^{\varepsilon'''}(x - y) dx dy.$$

However, since $\tilde{E}[\tilde{Y}_L(s, y)\varphi(s, y, \tilde{\omega})] \in L^2([0, t] \times \mathbb{S}_M)$, this tends to 0. This shows that $E[I^{(3, \varepsilon''')}\Phi]$ converges to 0 as $\varepsilon''' \downarrow 0$. \square

From Theorem 3.11 applied on $\tilde{\Omega}$, (3.48) and Lemma 3.16, the last term in the right hand side of (3.27) (on $\tilde{\Omega}$ and modulo $2L$, or (3.51) integrated with $\varphi(x)$) must converge weakly to a certain $\tilde{N}_L(t, x) = \tilde{N}_L(t, x; \tilde{\omega}, m)$ in $L^2(\tilde{\Omega} \times [-L, L], \tilde{\mathcal{F}}_T, \tilde{P} \otimes \pi_L)$; c.f. Remark 3.5. Thus we obtain the equation:

$$(3.54) \quad \tilde{Y}_L(t, x) = \tilde{Y}_L(0, x) + \frac{1}{2} \int_0^t \partial_x^2 \tilde{Y}_L(s, x) ds + \frac{1}{24} \int_0^t \tilde{Y}_L(s, x) ds$$

$$+ \int_0^t \tilde{Y}_L(s, x) d\tilde{W}(s, x) + \tilde{N}_L(t, x),$$

for a.e. $x \in (\text{supp } \rho)^c$, and a.e. $(t, \tilde{\omega}, m)$ in the limit.

Recalling the definition of $\tilde{\Omega}_L$, we see that the limit $\tilde{N}_L(t, x) = \tilde{N}_L(t, x; \tilde{\omega}, m)$ of the jump part of $\tilde{Y}_L^{\varepsilon'''}$ vanishes on $\tilde{\Omega}_L \times [-\frac{L}{2}, \frac{L}{2}]$, i.e., $\tilde{N}_L(t, x) = 0$ for all $t \in [0, T]$ on $\tilde{\Omega}_L \times [-\frac{L}{2}, \frac{L}{2}]$. Moreover, since $\tilde{P}(\cup_{L \in \mathbb{N}} \tilde{\Omega}_L) = 1$ holds, from (3.54) and (3.50), we see that $\tilde{Z}(t, x; \tilde{\omega})$ satisfies the SPDE (1.14) on $(\text{supp } \rho)^c$. However, choosing another $\tilde{\rho}$, which satisfies the same condition as ρ stated at the beginning of Section 3.2, such that $\text{supp } \rho \cap \text{supp } \tilde{\rho} = \emptyset$, we see that $\tilde{Z}(t, x; \tilde{\omega})$ satisfies the SPDE (1.14) on the whole torus \mathbb{S}_M . Note that \tilde{Z} is defined independently of the choice of ρ .

Since the solution of (1.14) is unique and continuous in (t, x) , we find that $\tilde{Z}(t, x; \tilde{\omega})$ is continuous in (t, x) , \tilde{P} -a.s. In particular, $\tilde{Z}(T, x; \tilde{\omega})$ is consistent with $\tilde{h}(T, \cdot)$, which was introduced in Step 1 of this subsection. Since $\tilde{Z}^{\varepsilon'''}(T)$ converges to $\tilde{Z}(T)$ \tilde{P} -a.s. and the limit is uniquely characterized by the SPDE (1.14), we don't need to take the subsequences. This concludes the proof of Theorem 1.1.

3.6 Proof of Theorem 1.2

We first prove that the solution $Z^M(t, x), x \in \mathbb{R}$ of the SPDE (1.14) on \mathbb{S}_M periodically extended to the whole line \mathbb{R} weakly converges to the solution $Z(t, x), x \in \mathbb{R}$ of the SPDE (1.11):

Proposition 3.17. *Assume that $Z(0, \cdot) \in L_r^2(\mathbb{R}), r > 0$, is given and the initial value of Z^M is determined by $Z^M(0, x) = Z(0, x)$ for $|x| \leq M/2$ and periodically extended to \mathbb{R} . Then, we have the followings.*

- (1) $\{Z^M(t, x)\}_{M \geq 1}$ is tight on $C([0, \infty), L_r^2(\mathbb{R}))$.
- (2) $Z^M(t, x)$ weakly converges to $Z(t, x)$ on the space $C([0, \infty), L_r^2(\mathbb{R}))$ as $M \rightarrow \infty$.

To prove this proposition, we prepare a lemma. Recall that the fundamental solution p^M of the parabolic operator $\frac{\partial}{\partial t} - \frac{1}{2} \frac{\partial^2}{\partial x^2}$ on \mathbb{S}_M is given by

$$p^M(t, x, y) = \sum_{n=-\infty}^{\infty} p(t, x, y + nM), \quad x, y \in \mathbb{S}_M = [0, M).$$

We define a function $\chi_M(x), x \in \mathbb{R}$ by $\chi_M(x) = |x|$ for $|x| \leq \frac{M}{2}$ and then by periodically extending it on \mathbb{R} . The following lemma is an extension of Lemma 6.2 of [21] to \mathbb{S}_M ; see also [8].

Lemma 3.18. (1) *For every $0 < \delta < 1$ and $T > 0$, there exists $C = C_{\delta, T} > 0$ such that*

$$\int_0^{t'} \int_{\mathbb{S}_M} (p^M(t' - s, x', y) - 1_{\{s \leq t\}} p^M(t - s, x, y))^2 dy \leq C(|t - t'|^{1/2} + |x - x'|^{1-\delta}),$$

holds for $0 \leq t < t' \leq T, x, x' \in \mathbb{S}_M$.

- (2) *For every $r \in \mathbb{R}$ and $T > 0$,*

$$\sup_{M \geq 1} \sup_{0 \leq t \leq T} \sup_{x \in \mathbb{S}_M} e^{-r\chi_M(x)} \int_{\mathbb{S}_M} p^M(t, x, y) e^{r\chi_M(y)} dy < \infty.$$

Proof. To show (1), we first expand the square inside the integral in y and apply Chapman-Kolmogorov's equality for the integral of products of p^M . Then, (1) is shown by the following easily shown uniform bounds on p^M :

$$0 < p^M(t, x, y) \leq \frac{C}{\sqrt{t}}, \quad \left| \frac{\partial p^M}{\partial x}(t, x, y) \right| \leq \frac{C}{t},$$

for $0 < t \leq T$ with $C = C_T > 0$ which is independent of M . For (2), we note that

$$\chi_M(x) - |a| \leq \chi_M(x + a) \leq \chi_M(x) + |a|, \quad x, a \in \mathbb{R},$$

and therefore

$$\int_{\mathbb{S}_M} p^M(t, x, y) e^{r\chi_M(y)} dy = E \left[e^{r\chi_M(x+B(t))} \right] \leq e^{r\chi_M(x)} E \left[e^{|rB(t)|} \right] \leq C e^{r\chi_M(x)},$$

where $B(t)$ is a Brownian motion starting at 0. \square

Proof of Proposition 3.17. For (1), we follow the proof of Theorem 2.2 in [21]. The SPDE (1.14) can be rewritten into the mild form:

$$(3.55) \quad \begin{aligned} Z^M(t, x) &= \int_{\mathbb{S}_M} Z^M(0, y) p^M(t, x, y) dy \\ &+ \int_0^t ds \int_{\mathbb{S}_M} e^{\frac{1}{24}(t-s)} p^M(t-s, x, y) Z^M(s, y) W(ds dy). \end{aligned}$$

We first show that

$$\sup_{M \geq 1} \sup_{0 \leq t \leq T} \int_{\mathbb{S}_M} e^{-r\chi_M(x)} E[|Z^M(t, x)|^{2p}] dx < \infty,$$

for $p \geq 1$ and $T > 0$ by using Lemma 3.18-(2). Then, by Lemma 3.18-(1), we have the bound:

$$E[|X^M(t, x) - X^M(t', x')|^{2p}] \leq C\{|t - t'|^p + |x - x'|^{2p(1-\delta)}\},$$

for every $0 \leq t < t' \leq T$, $x, x' \in \mathbb{S}_M : |x - x'| \leq 1$, where $X^M(t, x)$ is the second term in the right hand side of (3.55). Since the first term is easily treated, this proves the tightness.

To show (2), we rely on the martingale formulation as in the proof of Theorem 2.1. We consider the operator

$$\mathcal{L}^Z \Phi(Z) = \int_{\mathbb{R}} \left(\frac{1}{2} Z^2(x) D^2 \Phi(x, x; Z) + \left(\frac{1}{2} \partial_x^2 Z(x) + \frac{1}{24} Z(x) \right) D \Phi(x; Z) \right) dx,$$

for $\Phi \in \mathcal{D}(\mathcal{C})$ and $\mathcal{L}_M^Z \Phi$ for $\Phi \in \mathcal{D}(\mathcal{C}_M)$ by replacing the integral over \mathbb{R} by \mathbb{S}_M . Then, the distribution P^M of $Z^M(\cdot, \cdot)$ is a solution of $(\mathcal{L}_M^Z, \mathcal{D}(\mathcal{C}_M))$ -martingale problem and $\{P^M\}$ is tight. Then, it is easy to see that any weak limit of P^M as $M \rightarrow \infty$ is a solution of $(\mathcal{L}^Z, \mathcal{D}(\mathcal{C}))$ -martingale problem. Since the limit is unique, this concludes the proof of (2). \square

Since ν^M (periodically extended on \mathbb{R}) weakly converges to ν as $M \rightarrow \infty$ on $L_r^2(\mathbb{R})$, $r > 1$ (note $E^\nu[e^{2B(x)}] = e^{2|x|}$, $x \in \mathbb{R}$), this proposition shows that ν is invariant for the tilt variable of the logarithm of the solution $Z(t, \cdot)$ of the SPDE (1.11).

Let $\bar{Z}(t)$ be the solution of the SPDE (1.11) and set

$$Z(t) := e^{-\frac{1}{24}t} \bar{Z}(t).$$

Then, one can easily see that $Z(t)$ is a solution of (1.3) and $Z(t) \sim \bar{Z}(t)$ by the definition. Since the tilt variable of $Z^\varepsilon(t)$ has $\tilde{\mu}^\varepsilon$ as invariant probability measure, we see in the limit the tilt variable of $\bar{Z}(t)$ has $\tilde{\mu}$ as invariant probability measure. Since $Z(t) \sim \bar{Z}(t)$, this concludes that the tilt variable of $Z(t)$ also has $\tilde{\mu}$ as invariant probability measure. In other words, we obtain the invariance of the distribution of the geometric Brownian motion for the tilt process determined by the stochastic heat equation (1.3):

Proposition 3.19. *For any bounded and continuous function $G = G(Z)$ on $\tilde{\mathcal{C}}_+$ and for any $\varepsilon > 0$, $t \geq 0$, we have that*

$$\int_{\tilde{\mathcal{C}}_+} G(Z(t)) d\tilde{\mu} = \int_{\tilde{\mathcal{C}}_+} G(Z(0)) d\tilde{\mu},$$

where $\tilde{\mu}$ is a probability distribution on $\tilde{\mathcal{C}}_+$ of $Z(\cdot) = e^{B(\cdot)}$ with $B(\cdot) \in \tilde{\mathcal{C}}$ distributed under ν .

In order to deduce Theorem 1.2 from Proposition 3.19, one needs to now recover the height at the origin $h(t, 0)$, which is defined by $h(t, x) = \log Z(t, x)$. However, since this does not really work as in Section 2.7, we consider its smooth approximation. Let η^ε be the function introduced previously, and define a function $h^\varepsilon(x, Z) := \log(Z * \eta^\varepsilon(x))$ of $x \in \mathbb{R}$ and $Z \in \mathcal{C}_+$. Let $Z(t)$ be the solution of the SPDE (1.3). Then, by Itô's formula, the approximated height $h^\varepsilon(t, x) := h^\varepsilon(x, Z(t))$ satisfies

$$(3.56) \quad h^\varepsilon(t, x) = h^\varepsilon(0, x) + \int_0^t b^\varepsilon(x, Z(s)) ds + \int_0^t \int_{\mathbb{R}} \sigma^\varepsilon(x, y, Z(s)) W(ds dy),$$

where

$$b^\varepsilon(x, Z) = \frac{1}{2} \left\{ \partial_x^2 h^\varepsilon(x, Z) + (\partial_x h^\varepsilon(x, Z))^2 - \frac{(Z^2 * (\eta^\varepsilon)^2)(x)}{(Z * \eta^\varepsilon(x))^2} \right\},$$

$$\sigma^\varepsilon(x, y, Z) = \frac{\eta^\varepsilon(x - y) Z(y)}{Z * \eta^\varepsilon(x)}.$$

The key point is that, as functions of Z , both b^ε and σ^ε are defined on the quotient space $\tilde{\mathcal{C}}_+$. Therefore, once $Z(t) \in \tilde{\mathcal{C}}_+$ is determined by solving the SPDE (1.3), we can recover its height as

$$(3.57) \quad h^\varepsilon(t, 0) = h^\varepsilon(0, 0) + X_t,$$

where X_t is the sum of the second and third terms in the right hand side of (3.56) with $x = 0$, which depends only on $Z(\cdot)$.

Lemma 3.20. *For any bounded, integrable and continuous function $G = G(h, Z)$ on $\mathcal{C}_+ \equiv \mathbb{R} \times \tilde{\mathcal{C}}_+$ and for any $\varepsilon > 0$, $t \geq 0$, we have that*

$$(3.58) \quad \int_{\mathbb{R} \times \tilde{\mathcal{C}}_+} G(h^\varepsilon(t, 0), Z(t)) dh_0 d\tilde{\mu} = \int_{\mathbb{R} \times \tilde{\mathcal{C}}_+} G(h^\varepsilon(0, 0), Z(0)) dh_0 d\tilde{\mu}.$$

Proof. From (3.57) and the translation-invariance of the Lebesgue measure, the left hand side of (3.58) is equal to

$$\int_{\mathbb{R} \times \tilde{\mathcal{C}}_+} G(h^\varepsilon(0, 0) + X_t, Z(t)) dh_0 d\tilde{\mu} = \int_{\mathbb{R} \times \tilde{\mathcal{C}}_+} G(h^\varepsilon(0, 0), Z(t)) dh_0 d\tilde{\mu},$$

by performing the integral in dh_0 first. But, this is equal to the right hand side of (3.58) by the invariance of μ under $Z(t) \in \tilde{\mathcal{C}}_+$; see Proposition 3.19. \square

Letting ε tend to zero in (3.58), we obtain (3.58) also for $(h(t, 0), Z(t))$. This implies the invariance of the product measure $dh_0 d\tilde{\mu}$ for this joint process with the state space $\mathbb{R} \times \tilde{\mathcal{C}}_+$. However, since the image measure of $dh_0 d\tilde{\mu}$ under the map $(h_0, Z) \in \mathbb{R} \times \tilde{\mathcal{C}}_+ \mapsto e^{h_0} Z \in \mathcal{C}_+$ is nothing but μ , the proof of Theorem 1.2 is completed.

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